

CONTACT STRUCTURES, SUTURED FLOER HOMOLOGY AND TQFT

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ABSTRACT. We describe the natural gluing map on sutured Floer homology which is induced by the inclusion of one sutured manifold (M', Γ') into a larger sutured manifold (M, Γ) , together with a contact structure on $M - M'$. As an application of this gluing map, we produce a $(1+1)$ -dimensional TQFT by dimensional reduction and study its properties.

1. INTRODUCTION

Since its inception around 2001, Ozsváth and Szabó's Heegaard Floer homology [OS1, OS2] has been developing at a breakneck pace. In one direction, Ozsváth-Szabó [OS4] and, independently, Rasmussen [Ra] defined knot invariants, called knot Floer homology, which categorified the Alexander polynomial. Although its initial definition was through Lagrangian Floer homology, knot Floer homology was recently shown to admit a completely combinatorial description by Manolescu-Ozsváth-Sarkar [MOS]. Knot Floer homology is a powerful invariant which detects the genus of a knot by the work of Ozsváth-Szabó [OS6], and detects fibered knots by the work of Ghiggini [Gh] and Ni [Ni]. (The latter was formerly called the “fibered knot conjecture” of Ozsváth-Szabó).

One of the offshoots of the effort to prove this fibered knot conjecture is the definition of a relative invariant for a 3-manifold with boundary. In a pair of important papers [Ju1, Ju2], András Juhász generalized the hat versions of Ozsváth and Szabó's Heegaard Floer homology [OS1, OS2] and link Floer homology [OS4] theories, and assigned a Floer homology group $SFH(M, \Gamma)$ to a balanced sutured manifold (M, Γ) . (A related theory is being worked out by Lipshitz [Li1, Li2] and Lipshitz-Ozsváth-Thurston [LOT].)

In [HKM3], the present authors defined an invariant $EH(M, \Gamma, \xi)$ of (M, Γ, ξ) , a contact 3-manifold (M, ξ) with convex boundary and dividing set Γ on ∂M , as an element in $SFH(-M, -\Gamma)$. Our invariant generalized the contact class in Heegaard Floer homology in the closed case, as defined by Ozsváth and Szabó [OS3] and reformulated by the authors in [HKM2]. The definition of the contact invariant was made possible by the work of Giroux [Gi2], which provides special Morse functions (called *convex Morse functions*) or, equivalently, open book decompositions which are adapted to contact structures.

Recall that a *sutured manifold* (M, Γ) , due to Gabai [Ga], is a compact, oriented, not necessarily connected 3-manifold M with boundary, together with an oriented embedded 1-manifold $\Gamma \subset \partial M$

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which bounds a subsurface of ∂M .¹ More precisely, there is an open subsurface $R_+(\Gamma) \subset \partial M$ (resp. $R_-(\Gamma)$) on which the orientation agrees with (resp. is the opposite of) the orientation on ∂M induced from M , and $\Gamma = \partial R_+(\Gamma) = \partial R_-(\Gamma)$ as oriented 1-manifolds. A sutured manifold (M, Γ) is *balanced* if M has no closed components, $\pi_0(\Gamma) \rightarrow \pi_0(\partial M)$ is surjective, and $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$ on the boundary of every component of M . In particular, every boundary component of ∂M nontrivially intersects the suture Γ .

In this paper, we assume that all sutured manifolds are balanced and all contact structures are cooriented. Although every connected component of a balanced sutured manifold (M, Γ) must have nonempty boundary, our theorems are also applicable to closed, oriented, connected 3-manifolds M . Following Juhász [Ju1], a closed M can be replaced by a balanced sutured manifold as follows: Let B^3 be a 3-ball inside M , and consider $M - B^3$. On $\partial(M - B^3) = S^2$, let $\Gamma = S^1$. Since $SFH(M - \text{int}(B^3), \Gamma = S^1)$ is naturally isomorphic to $\widehat{HF}(M)$, we can view a closed M as $(M - \text{int}(B^3), S^1)$.

The goal of this paper is to understand the effect of cutting/gluing of sutured manifolds. We first define a map which is induced from the inclusion of one balanced sutured manifold (M', Γ') into another balanced sutured manifold (M, Γ) , in the presence of a “compatible” contact structure ξ on $M - \text{int}(M')$. Here we say that (M', Γ') is a *sutured submanifold* of a sutured manifold (M, Γ) if M' is a submanifold with boundary of M , so that $M' \subset \text{int}(M)$. If a connected component N of $M - \text{int}(M')$ contains no components of ∂M we say that N is *isolated*.

We will work with Floer homology groups over the ring \mathbb{Z} . With \mathbb{Z} -coefficients, the contact invariant $EH(M, \Gamma, \xi)$ is a subset of $SFH(-M, -\Gamma)$ of cardinality 1 or 2 of type $\{\pm x\}$, where $x \in SFH(-M, -\Gamma)$. (The cardinality is 1 if and only if x is a 2-torsion element.) Over $\mathbb{Z}/2\mathbb{Z}$, the ± 1 ambiguity disappears, and $EH(M, \Gamma, \xi) \in SFH(-M, -\Gamma)$.

The following theorem is the main technical result of our paper.

Theorem 1.1. *Let (M', Γ') be a sutured submanifold of (M, Γ) , and let ξ be a contact structure on $M - \text{int}(M')$ with convex boundary and dividing set Γ on ∂M and Γ' on $\partial M'$. If $M - \text{int}(M')$ has m isolated components, then ξ induces a natural map:*

$$\Phi_\xi : SFH(-M', -\Gamma') \rightarrow SFH(-M, -\Gamma) \otimes V^{\otimes m},$$

which is well-defined only up to an overall \pm sign. Moreover,

$$\Phi_\xi(EH(M', \Gamma', \xi')) = EH(M, \Gamma, \xi' \cup \xi) \otimes (x \otimes \cdots \otimes x),$$

where x is the contact class of the standard tight contact structure on $S^1 \times S^2$ and ξ' is any contact structure on M' with boundary condition Γ' . Here $V = \widehat{HF}(S^1 \times S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$ is a \mathbb{Z} -graded vector space where the two summands have grading which differ by one, say 0 and 1.

The choice of a contact structure ξ on $M - M'$ plays a key role as the “glue” or “field” which takes classes in $SFH(-M', -\Gamma')$ to classes in $SFH(-M, -\Gamma) \otimes V^{\otimes m}$. We emphasize that the gluing map Φ_ξ is usually not injective. The statement of the theorem, in particular the “naturality” and the V factor, will be explained in more detail in Section 3.

One immediate corollary of Theorem 1.1 is the following result, essentially proved in [HKM3]:

Corollary 1.2. *Let $i : (M', \Gamma', \xi') \rightarrow (M, \Gamma, \xi)$ be an inclusion such that $\xi|_{M'} = \xi'$. If $EH(M, \Gamma, \xi) \neq 0$, then $EH(M', \Gamma', \xi') \neq 0$.*

¹This definition is slightly different from that of Gabai [Ga].

Gluing along convex surfaces. Specifying a suture Γ on ∂M is equivalent to prescribing a translation-invariant contact structure $\zeta_{\partial M}$ in a product neighborhood of ∂M with dividing set Γ . Let U be a properly embedded surface of (M, Γ) satisfying the following:

- There exists an invariant contact structure ζ_U , defined in a neighborhood of U , which agrees with $\zeta_{\partial M}$ near ∂U ;
- U is convex with possibly empty Legendrian boundary and has a dividing set Γ_U with respect to ζ_U .

Let (M', Γ') be the sutured manifold obtained by cutting (M, Γ) along U and edge-rounding. (See [H1] for a description of edge-rounding.) By slightly shrinking M' , we obtain the tight contact structure $\zeta = \zeta_{\partial M} \cup \zeta_U$ on $M - \text{int}(M')$. The contact structure ζ induces the map

$$\Phi_\zeta : SFH(-M', -\Gamma') \rightarrow SFH(-M, -\Gamma) \otimes V^{\otimes m},$$

for an appropriate m .

Summarizing, we have the following:

Theorem 1.3 (Gluing Map). *Let (M', Γ') be a sutured manifold and let U_+ and U_- be disjoint subsurfaces of $\partial M'$ (with the orientation induced from $\partial M'$) which satisfy the following:*

- (1) *Each component of ∂U_\pm transversely and nontrivially intersects Γ' .*
- (2) *There is an orientation-reversing diffeomorphism $\phi : U_+ \rightarrow U_-$ which takes $\Gamma'|_{U_+}$ to $\Gamma'|_{U_-}$ and takes $R_\pm(U_+)$ to $R_\mp(U_-)$.*

Let (M, Γ) be the sutured manifold obtained by gluing U_+ and U_- via ϕ , and smoothing. Then there is a natural gluing map

$$\Phi : SFH(-M', -\Gamma') \rightarrow SFH(-M, -\Gamma) \otimes V^{\otimes m},$$

where m equals the number of components of U_+ that are closed surfaces. Moreover, if (M, Γ, ξ) is obtained from (M', Γ', ξ') by gluing, then

$$\Phi(EH(M', \Gamma', \xi')) = EH(M, \Gamma, \xi) \otimes (x \otimes \cdots \otimes x),$$

where x is the contact class of the standard tight contact structure on $S^1 \times S^2$.

In particular, when Γ_U is ∂ -parallel, i.e., each component of Γ_U cuts off a half-disk which intersects no other component of Γ_U , then the convex decomposition $(M, \Gamma) \xrightarrow{(U, \Gamma_U)} (M', \Gamma')$ corresponds to a sutured manifold decomposition by [HKM1]. In Section 6 we indicate why our gluing map

$$\Phi : SFH(-M', -\Gamma') \hookrightarrow SFH(-M, -\Gamma)$$

is the same as the direct summand map constructed in [HKM3, Section 6].²

(1 + 1)-dimensional TQFT. We now describe a (1 + 1)-dimensional TQFT, which is obtained by dimensional reduction of sutured Floer homology and gives an invariant of multicurves on surfaces. (In this paper we loosely use the terminology “TQFT”. The precise properties satisfied by our “TQFT” are given in Section 7.)

Let Σ be a compact, oriented surface with nonempty boundary $\partial\Sigma$, and F be a finite set of points of $\partial\Sigma$, where the restriction of F to each component of $\partial\Sigma$ consists of an even number ≥ 2

²In [Ju2], Juhász proves that a sutured manifold gluing induces a direct summand map $SFH(-M', -\Gamma') \hookrightarrow SFH(-M, -\Gamma)$. Although it is expected that this map agrees with the natural gluing map, this has not been proven.

of points. Moreover, the connected components of $\partial\Sigma - F$ are alternately labeled $+$ and $-$. Also let K be a properly embedded, oriented 1-dimensional submanifold of Σ whose boundary is F and which divides Σ into R_+ and R_- in a manner compatible with the labeling of $\partial\Sigma - F$. Let ξ_K be the S^1 -invariant contact structure on $S^1 \times \Sigma$ which traces the dividing set K on each $\{pt\} \times \Sigma$. Let $F_0 \subset \partial\Sigma$ be obtained from F by shifting slightly in the direction of $\partial\Sigma$. The corresponding contact invariant $EH(\xi_K)$ is a subset of $SFH(-(S^1 \times \Sigma), -(S^1 \times F_0))$ of the form $\{\pm x\}$. The TQFT assigns to each (Σ, F) a graded \mathbb{Z} -module $V(\Sigma, F) = SFH(-(S^1 \times \Sigma), -(S^1 \times F_0))$ and to each K the subset $EH(\xi_K) \subset V(\Sigma, F)$.

One application of the TQFT is the following:

Theorem 1.4. *The contact invariant in sutured Floer homology does not always admit a single-valued representative with \mathbb{Z} -coefficients.*

Next, we say that K is *isolating* if $\Sigma - K$ contains a component that does not intersect $\partial\Sigma$. Using the TQFT properties we will prove:

Theorem 1.5. *Over $\mathbb{Z}/2\mathbb{Z}$, $EH(\xi_K) \neq 0$ if and only if K is nonisolating.*

Theorem 1.5, combined with Corollary 1.2, expands our repertoire of universally tight contact structures which are not embeddable in Stein fillable contact 3-manifolds.

Corollary 1.6. *Let ξ_K be the S^1 -invariant contact structure on $S^1 \times \Sigma$ corresponding to the dividing set $K \subset \Sigma$. Then ξ_K cannot be embedded in a Stein fillable (or strongly symplectically fillable) closed contact 3-manifold if K is isolating.*

Finally, we remark that $V(\Sigma, F)$ is the Grothendieck group of a category $\mathcal{C}(\Sigma, F)$, called the *contact category*, whose objects are dividing sets on (Σ, F) and whose morphisms are contact structures on $\Sigma \times [0, 1]$. The contact category will be treated in detail in [H3].

Organization of the paper. In Section 2 we review the notions of sutured Floer homology and partial open book decompositions, which appeared in [Ju1, Ju2, HKM3]. Section 3 is devoted to explaining Theorem 1.1, in particular the V factor and the naturality statement. Theorem 1.1 will be proved in Sections 4 and 5. The map Φ_ξ will be defined in Section 4 and the fact that Φ_ξ is a natural map will be proved in Section 5. We remark that, although the basic idea of the definition of Φ_ξ is straightforward, the actual definition and the proof of naturality are unfortunately rather involved. Basic properties of the gluing map will be given in Section 6. Section 7 is devoted to analyzing the $(1 + 1)$ -dimensional TQFT.

2. PRELIMINARIES

We first review some notions which appeared in [Ju1, Ju2] and [HKM3].

Let (M, Γ) be a balanced sutured manifold. Then a *Heegaard splitting* (Σ, α, β) for (M, Γ) consists of a properly embedded oriented surface Σ in M with $\partial\Sigma = \Gamma$ and two sets of disjoint simple closed curves $\alpha = \{\alpha_1, \dots, \alpha_r\}$ and $\beta = \{\beta_1, \dots, \beta_r\}$. The Heegaard surface Σ compresses to $R_-(\Gamma)$ along the collection α and to $R_+(\Gamma)$ along the collection β . The number of α curves equals the number of β curves since (M, Γ) is assumed to be balanced.

To define the sutured Floer Homology groups, as introduced by Juhász, we consider the Lagrangian tori $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_r$ and $\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_r$ in $Sym^r(\Sigma)$. Let $CF(\Sigma, \alpha, \beta)$ be the free

\mathbb{Z} -module generated by the points $\mathbf{x} = (x_1, \dots, x_r)$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. In the definition of the boundary map for sutured Floer homology, the suture Γ plays the role of the basepoint. Denote by $\mathcal{M}_{\mathbf{x}, \mathbf{y}}$ the 0-dimensional (after quotienting by the natural \mathbb{R} -action) moduli space of holomorphic maps u from the unit disk $D^2 \subset \mathbb{C}$ to $Sym^r(\Sigma)$ that (i) send $1 \mapsto \mathbf{x}$, $-1 \mapsto \mathbf{y}$, $S^1 \cap \{\text{Im } z \geq 0\}$ to \mathbb{T}_α and $S^1 \cap \{\text{Im } z \leq 0\}$ to \mathbb{T}_β , and (ii) avoid $\partial\Sigma \times Sym^{r-1}(\Sigma) \subset Sym^r(\Sigma)$. Then define

$$\partial\mathbf{x} = \sum_{\mu(\mathbf{x}, \mathbf{y})=1} \#(\mathcal{M}_{\mathbf{x}, \mathbf{y}}) \mathbf{y},$$

where $\mu(\mathbf{x}, \mathbf{y})$ is the relative Maslov index of the pair and $\#(\mathcal{M}_{\mathbf{x}, \mathbf{y}})$ is a signed count of points in $\mathcal{M}_{\mathbf{x}, \mathbf{y}}$. The homology of $CF(\Sigma, \alpha, \beta)$ is the sutured Floer homology group $SFH(\Sigma, \alpha, \beta) = SFH(M, \Gamma)$.

In [HKM3], the present authors defined an invariant $EH(M, \Gamma, \xi)$ of (M, Γ, ξ) , a contact 3-manifold with convex boundary and dividing set Γ on ∂M , as an element in $SFH(-M, -\Gamma)$. This invariant generalizes the contact class in Heegaard Floer homology in the closed case, as defined by Ozsváth and Szabó [OS3], and described from a different point of view in [HKM2]. For the sake of completeness, we sketch the definition of the invariant $EH(M, \Gamma, \xi)$.

First consider the case when ξ is a contact structure on a closed manifold M . In [HKM2] we used an open book decomposition compatible with ξ to construct a convenient Heegaard decomposition (Σ, α, β) for M in which the contact class was a distinguished element in $\widehat{HF}(\Sigma, \alpha, \beta)$. Recall that an open book decomposition for M is a pair (S, h) consisting of a surface S with boundary and a homeomorphism $h : S \xrightarrow{\sim} S$ with $h|_{\partial S} = id$, so that $M \simeq S \times [0, 1] / \sim_h$, where $(x, 1) \sim_h (h(x), 0)$ for $x \in S$ and $(x, t) \sim_h (x, t')$ for $x \in \partial S$, $t, t' \in [0, 1]$. A Heegaard decomposition (Σ, β, α) for $-M$ (recall that the contact class lives in the Heegaard Floer homology of $-M$) is obtained from the two handlebodies $H_1 = S \times [0, \frac{1}{2}] / \sim_h$ and $H_2 = S \times [\frac{1}{2}, 1] / \sim_h$, which are glued along the common boundary $\Sigma = (S \times \{\frac{1}{2}\}) \cup -(S \times \{0\})$ by $id \cup h$. Take a family of properly embedded disjoint arcs a_i that cuts the surface S into a disk, and small push-offs b_i of a_i (in the direction of the boundary) such that b_i intersects a_i in exactly one point. The compressing disks for H_1 and H_2 , respectively, are $D_{a_i} = a_i \times [0, \frac{1}{2}]$ and $D_{b_i} = b_i \times [\frac{1}{2}, 1]$; set $\alpha_i = \partial D_{a_i}$ and $\beta_i = \partial D_{b_i}$. We call the family of arcs a_i a *basis* for S , and show in [HKM2] that the element of Heegaard Floer homology that corresponds to the generator $\mathbf{x} = (x_1, \dots, x_n)$, where x_i is the unique intersection point of $a_i \times \{\frac{1}{2}\}$ and $b_i \times \{\frac{1}{2}\}$, is independent of the choice of basis for S and the compatible open book decomposition. Moreover, it is the contact class defined by Ozsváth and Szabó.

To define the contact class $EH(M, \Gamma, \xi)$ in the case of a balanced sutured manifold, we generalize the notions of an “open book” and a “basis”, involved in the definition of the contact invariant above. Let (A, B) be a pair consisting of a surface A with nonempty boundary and a subsurface $B \subset A$. A collection $\{a_1, \dots, a_k\}$ of properly embedded disjoint arcs in A is called a *basis* for (A, B) if each a_i is disjoint from B and $A - \bigcup_{i=1}^k a_i$ deformation retracts to B . A *partial open book* $(S, R_+(\Gamma), h)$ consists of the following data: a compact, oriented surface S with nonempty boundary, a subsurface $R_+(\Gamma) \subset S$, and a “partial” monodromy map $h : P \rightarrow S$, where $P \subset S$ is the closure of $S - \overline{R_+(\Gamma)}$ and $h(x) = x$ for all $x \in (\partial S) \cap P$. We say that $(S, R_+(\Gamma), h)$ is a *partial open book decomposition* for (M, Γ) if $M \simeq S \times [0, 1] / \sim_h$, where the equivalence relation is $(x, 1) \sim_h (h(x), 0)$ for $x \in P$ and $(x, t) \sim_h (x, t')$ for $x \in \partial S$, $t, t' \in [0, 1]$. Since the monodromy h is defined only on P , the space obtained after gluing has boundary consisting of $R_+ \times \{1\}$ and

$R_- \times \{0\}$, where $R_- = S - h(P)$. The suture Γ is the common boundary of $R_+ \times \{1\}$ and $R_- \times \{0\}$.

To see a handlebody decomposition of M from this point of view, let $H_1 = S \times [0, \frac{1}{2}] / \sim$, where $(x, t) \sim (x, t')$ if $x \in \partial S$ and $t, t' \in [0, \frac{1}{2}]$ and let $H_2 = P \times [\frac{1}{2}, 1] / \sim$, where $(x, t) \sim (x, t')$ if $x \in \partial P$ and $t, t' \in [\frac{1}{2}, 1]$. It is clear that we can think of $M \simeq S \times [0, 1] / \sim_h$ as $M \simeq H_1 \cup H_2 / \text{gluing}$, where the handlebodies are glued along portions of their boundary as follows: $(x, \frac{1}{2}) \in H_1$ is identified to $(x, \frac{1}{2}) \in H_2$ and $(x, 1) \in H_2$ is identified with $(h(x), 0) \in H_1$ for $x \in P$. This leaves $R_+ \times \{\frac{1}{2}\}$ and $R_- \times \{0\}$ as the boundary of the identification space. Now let $\{a_1, \dots, a_k\}$ be a *basis* for $(S, R_+(\Gamma))$ in the sense defined above. Let $b_i, i = 1, \dots, k$, be pushoffs of a_i in the direction of ∂S so that a_i and b_i intersect exactly once. Then it is not hard to see that if we set $\Sigma = (S \times \{0\}) \cup (P \times \{\frac{1}{2}\})$, $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ and $\beta_i = (b_i \times \{\frac{1}{2}\}) \cup (h(b_i) \times \{0\})$, then (Σ, β, α) is a Heegaard diagram for $(-M, -\Gamma)$.

The two handlebodies H_1 and H_2 defined above by the open book decomposition $(S, R_+(\Gamma), h)$ carry unique product disk decomposable contact structures. After gluing, they determine a contact structure $\xi_{(S, R_+(\Gamma), h)}$ on (M, Γ) . We say that a partial open book decomposition $(S, R_+(\Gamma), h)$ and a contact structure ξ are *compatible* if $\xi = \xi_{(S, R_+(\Gamma), h)}$. On the other hand, as in the closed manifold case, every contact structure ξ with convex boundary on a sutured manifold (M, Γ) gives rise to a compatible partial open book decomposition $(S, R_+(\Gamma), h)$.

3. EXPLANATION OF THEOREM 1.1

3.1. Naturality. We now explain what we mean by a “natural map” Φ_ξ . Recall the following theorem from Ozsváth-Szabó [OS7]:

Theorem 3.1 (Ozsváth-Szabó). *Given two Heegaard decompositions (Σ, α, β) , $(\bar{\Sigma}, \bar{\alpha}, \bar{\beta})$ of a closed 3-manifold M , the isomorphism*

$$\Psi : \widehat{HF}(\Sigma, \alpha, \beta) \xrightarrow{\sim} \widehat{HF}(\bar{\Sigma}, \bar{\alpha}, \bar{\beta}),$$

given as the composition of stabilization/destabilization, handleslide, and isotopy maps, is well-defined up to an overall factor of ± 1 and does not depend on the particular sequence chosen from (Σ, α, β) to $(\bar{\Sigma}, \bar{\alpha}, \bar{\beta})$.

This lack of monodromy allows us to “naturally” identify the isomorphic Heegaard Floer homology groups $\widehat{HF}(\Sigma, \alpha, \beta)$ and $\widehat{HF}(\bar{\Sigma}, \bar{\alpha}, \bar{\beta})$, up to an overall sign. Sutured Floer homology enjoys the same naturality property, that is, the isomorphism

$$\Psi : SFH(\Sigma, \alpha, \beta) \xrightarrow{\sim} SFH(\bar{\Sigma}, \bar{\alpha}, \bar{\beta})$$

is also well-defined up to an overall factor of ± 1 and is independent of the same type of choices if (Σ, α, β) , $(\bar{\Sigma}, \bar{\alpha}, \bar{\beta})$ are two Heegaard decompositions for (M, Γ) .

Next, suppose $(\Sigma', \beta', \alpha')$, $(\bar{\Sigma}', \bar{\beta}', \bar{\alpha}')$ are Heegaard splittings for $(-M', -\Gamma')$ and (Σ, β, α) , $(\bar{\Sigma}, \bar{\beta}, \bar{\alpha})$ are their extensions to $(-M, -\Gamma)$. We will restrict ourselves to working with a certain subclass of Heegaard splittings of $(-M', -\Gamma')$, namely those that are contact-compatible on a neighborhood of $\partial M'$, with respect to an invariant contact structure ζ which induces the dividing set Γ' on $\partial M'$. The Heegaard splittings for $(-M, -\Gamma)$ we will use extend those of $(-M', -\Gamma')$ and are contact-compatible with respect to ξ on $M - M'$. Assume $M - \text{int}(M')$ has no isolated

components. Then, in the statement of Theorem 1.1, we take the commutativity of the following diagram to be the definition of the *naturality* of Φ_ξ :

$$(3.1.1) \quad \begin{array}{ccc} SFH(\Sigma', \beta', \alpha') & \xrightarrow{(\Phi_\xi)_1} & SFH(\Sigma, \beta, \alpha) \\ \Psi_1 \downarrow & & \downarrow \Psi_2 \\ SFH(\overline{\Sigma}', \overline{\beta}', \overline{\alpha}') & \xrightarrow{(\Phi_\xi)_2} & SFH(\overline{\Sigma}, \overline{\beta}, \overline{\alpha}) \end{array}$$

Here the vertical maps Ψ_1, Ψ_2 are the natural isomorphisms of Theorem 3.1 and $(\Phi_\xi)_1, (\Phi_\xi)_2$ are the maps induced by ξ , to be defined in Section 4.

3.2. Explanation of the V factor. Consider (M, Γ, ξ) and a compatible partial open book decomposition $(S, R_+(\Gamma), h)$. Let $\{a_1, \dots, a_k\}$ be a basis for $(S, R_+(\Gamma))$. Consider a larger collection $\{a_1, \dots, a_k, a_{k+1}, \dots, a_{k+l}\}$ of properly embedded disjoint arcs in S which satisfy $a_i \subset P$, so that $S - \cup_{i=1}^{k+l} a_i$ is a disjoint union of disks D_j , $j = 1, \dots, l$, and a surface that deformation retracts to $R_+(\Gamma)$. For each j , pick $z_j \in D_j$ and consider a small neighborhood $N(z_j) \subset D_j$. Then $\{a_1, \dots, a_{k+l}\}$ becomes a basis for $(S, R_+(\Gamma) \cup (\cup_{j=1}^l N(z_j)))$. The Heegaard surface for $(S, R_+(\Gamma))$ is $\Sigma = (P \times \{1\}) \cup (S \times \{0\})$, whereas the Heegaard surface for $(S, R_+(\Gamma) \cup (\cup_{j=1}^l N(z_j)))$ is $\Sigma' = \Sigma - \cup_{j=1}^l N(z_j)$. As in Section 2, the a_i determine arcs b_i as well as closed curves α_i, β_i . We refer to the procedure of adding extra arcs to a basis for $(S, R_+(\Gamma))$ and extra $N(z_j)$'s to $R_+(\Gamma)$ as “placing extra dots” or “placing extra z_j 's”.

Claim. *The effect of placing an extra dot on $(S, R_+(\Gamma))$ on sutured Floer homology is that of taking the tensor product with $\widehat{HF}(S^1 \times S^2) \simeq V$.*

Proof. Consider the following situation: Suppose $\{a_1, \dots, a_k\}$ is a basis for $(S, R_+(\Gamma))$. Then add an extra properly embedded arc $a_{k+1} \subset P$ of S which is disjoint from a_1, \dots, a_k and such that one component D_1 of $S - a_{k+1}$ is a half-disk which is contained in P . Also add an extra dot z_1 in the component D_1 . The α_{k+1} and β_{k+1} corresponding to a_{k+1} intersect in exactly two points, and do not interact with the other α_i and β_i . By the placement of the extra dot,

$$\begin{aligned} & SFH(\Sigma', \{\beta_1, \dots, \beta_{k+1}\}, \{\alpha_1, \dots, \alpha_{k+1}\}) \\ & \simeq SFH(\Sigma, \{\beta_1, \dots, \beta_k\}, \{\alpha_1, \dots, \alpha_k\}) \otimes \widehat{HF}(S^1 \times S^2). \end{aligned}$$

Next, after a sequence of arc slides as in Section 3.1 of [HKM2] (or just handleslides), we can pass between any two bases of $(S, R_+(\Gamma) \cup N(z_1))$, where $N(z_1)$ is a small disk in $S - R_+(\Gamma)$ about z_1 . Since SFH is invariant under any sequence of handleslides, the claim follows. \square

Now we explain the V factors that appear in Theorem 1.1. Let us consider a partial open book decomposition (S', h') for any contact structure ξ' which is compatible with (M', Γ') , and let (S, h) be a partial open book decomposition for $\xi \cup \xi'$ which extends (S', h') . If no connected component of $M - \text{int}(M')$ is isolated, then a basis $\{a'_1, \dots, a'_k\}$ for (S', h') easily extends to a basis for (S, h) . If there are m isolated components of $M - \text{int}(M')$, then $S - \cup_{i=1}^k a'_i$ has m connected components which do not intersect $R_+(\Gamma)$ (and hence can never be completed to a basis for (S, h)). Instead, by adding m extra dots z_1, \dots, z_m , we can extend $\{a'_1, \dots, a'_k\}$ to a basis for $(S, R_+(\Gamma) \cup (\cup_{i=1}^m N(z_i)))$.

4. DEFINITION OF THE MAP Φ_ξ

In this section we define the chain map:

$$\Phi_\xi : CF(-M', -\Gamma') \rightarrow CF(-M, -\Gamma),$$

which induces the map, also called Φ_ξ by slight abuse of notation, on the level of homology. Let us assume that $M - \text{int}(M')$ has no isolated components. The general case follows without additional effort, by putting extra dots.

Sketch of the construction. We start by giving a quick overview of the construction of Φ_ξ . The actual definition needed to prove naturality is considerably more complicated and occupies the remainder of the section.

Let us first decompose $M = M' \cup M''$, where $M'' = M - \text{int}(M')$. Let Σ' be a Heegaard surface for the sutured manifold (M', Γ') . By definition, $\Sigma' \cap \partial M' = \Gamma'$. Next choose compressing disks α', β' on Σ' . Also let Σ'' be a Heegaard surface for the sutured manifold $(M'', \Gamma'' \cup -\Gamma')$.

Although it might appear natural to take the union of Σ' and Σ'' along their common boundary Γ' to create a Heegaard surface for M , we are presented with a problem. If we glue M' and M'' to obtain M , then, on the common boundary of M' and M'' , $R_\pm(\Gamma')$ from M' is glued to $R_\mp(-\Gamma')$ from M'' . As a result, the α -curves for Σ' and β -curves for Σ'' will be paired, and the β -curves for Σ' and α -curves for Σ'' will be paired, and we will be mixing homology and cohomology. A way around this problem is to insert the layer $N = T \times [0, 1]$, where $T = \partial M'$, so that $M = M' \cup N \cup M''$, $M' \cap N = T \times \{0\}$, and $M'' \cap N = T \times \{1\}$. Let Σ_N be a Heegaard surface for $(N, (-\Gamma' \times \{0\}) \cup (\Gamma' \times \{1\}))$. Then $\Sigma = \Sigma' \cup \Sigma_N \cup \Sigma''$ is a Heegaard surface for M .

A second issue which arises is that the union of compressing disks for M' , N , and M'' is not sufficient to give a full set of compressing disks for M . Our remedy is to use the contact invariant: First we take $(\Sigma', \beta', \alpha')$ to be *contact-compatible* near $\partial M'$. Roughly speaking, this means that $(\Sigma', \beta', \alpha')$, near $\partial M'$, looks like a Heegaard decomposition arising from a partial open book decomposition of a contact structure ζ which is defined near $\partial M'$ and has dividing set Γ' on $\partial M'$. Let Σ'' be a Heegaard surface which is compatible with $\xi|_{M''}$ and let Σ_N be a Heegaard surface which is compatible with the $[0, 1]$ -invariant contact structure $\xi|_N$. We then extend α' and β' by adding α'' and β'' which are compatible with $\xi \cup \zeta$, and then define

$$\Phi_\xi : CF(\Sigma', \beta', \alpha') \rightarrow CF(\Sigma, \beta' \cup \beta'', \alpha' \cup \alpha''),$$

$$\mathbf{y} \mapsto (\mathbf{y}, \mathbf{x}''),$$

where \mathbf{x}'' is the contact class $EH(\xi \cup \zeta)$, consisting of a point from each $\beta''_i \cap \alpha''_i$. □

We now give precise definitions. Let $T = \partial M'$ and let $T \times [-1, 1]$ be a neighborhood of $T = T \times \{0\}$ with a $[-1, 1]$ -invariant contact structure ζ which satisfies the following:

- $T_t = T \times \{t\}$, $t \in [-1, 1]$, are convex surfaces with dividing set $\Gamma' \times \{t\}$;
- $T \times [-1, 0] \subset M'$ and $T \times [0, 1] \subset M - \text{int}(M')$;
- $\xi|_{T \times [0, 1]} = \zeta|_{T \times [0, 1]}$.

In order to define Φ_ξ , we need to construct a suitable Heegaard splitting $(\Sigma', \beta', \alpha')$ for the sutured manifold $(-M', -\Gamma')$ and a contact-compatible extension to (Σ, β, α) for $(-M, -\Gamma)$. This will be done in several steps.

Step 1: Construction of $(\Sigma', \beta', \alpha')$. In this step we construct $(\Sigma', \beta', \alpha')$ which is *contact-compatible with respect to ζ near $\partial M'$* . (Although a little unwieldy, we take the construction below as the *definition* of a contact-compatible $(\Sigma', \beta', \alpha')$ with respect to ζ near $\partial M'$.) The technique is similar to the proof of Theorem 1.1 of [HKM3].

Let $0 < \varepsilon' < 1$. Start by choosing a cellular decomposition of $T_{-\varepsilon'}$ so that the following hold:

- The 1-skeleton K'_0 is Legendrian;
- Each edge of the cellular decomposition lies on the boundary of two distinct 2-cells Δ, Δ' ;
- The boundary of each 2-cell Δ intersects the dividing set $\Gamma_{T_{-\varepsilon'}}$ exactly twice.

Here we use the Legendrian realization principle and isotop $T_{-\varepsilon'}$, if necessary. Let K'_1 be a finite collection of Legendrian segments $\{p\} \times [-\varepsilon', 0]$, so that every endpoint $(p, -\varepsilon')$ in $T_{-\varepsilon'}$ lies in $K'_0 \cap (\Gamma' \times \{-\varepsilon'\})$ and for each connected component γ of Γ' there are at least two p 's in γ . Now let K'_2 be a graph attached to K'_0 so that $K'_0 \cup K'_2$ is a 1-skeleton of a cellular decomposition of $M' - (T \times (-\varepsilon', 0])$ and $\text{int}(K'_2) \subset M' - (T \times [-\varepsilon', 0])$. The graph K'_2 is obtained without reference to any contact structure. If we set $K' = K'_0 \cup K'_1 \cup K'_2$, then $\partial N(K')$ is the union of the tubular portion U and small disks $D_1, \dots, D_s \subset \partial M'$. Here $N(G)$ denotes the tubular neighborhood of a graph G . See Figure 1.

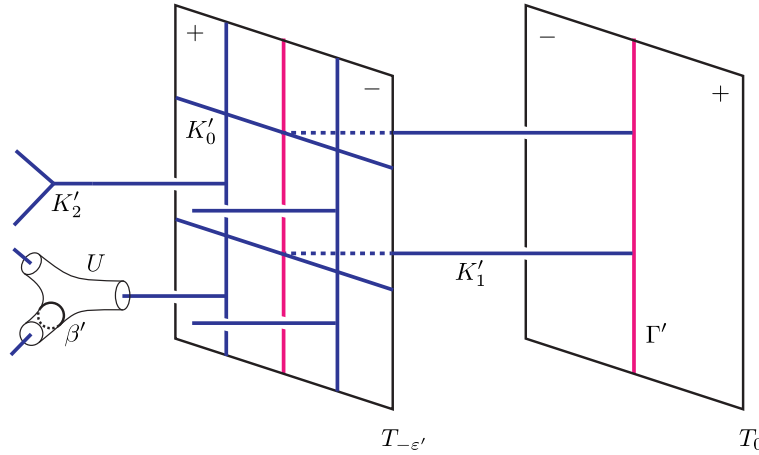


FIGURE 1.

Define the Heegaard surface Σ' to be (a surface isotopic to) the union $(R_-(\Gamma') - \cup_i D_i) \cup U$. Also modify $R_+(\Gamma')$ slightly so that $R_+(\Gamma') - \cup_i D_i$ is the new $R_+(\Gamma')$. The β' -curves are meridians (= boundaries of compressing disks) of $N(K')$, and the α' -curves are meridians of the complement, as chosen below.

After a contact isotopy, we may take the standard contact neighborhood $N(K'_0)$ to be $T \times [-\frac{3\varepsilon'}{2}, -\frac{\varepsilon'}{2}]$ with standard neighborhoods of Legendrian arcs of type $\{q\} \times [-\frac{3\varepsilon'}{2}, -\frac{\varepsilon'}{2}]$, $q \in \Gamma'$, removed. Now define the following decomposition of $T \times [-\frac{3\varepsilon'}{2}, 0]$ into two handlebodies:

$$H'_1 = (T \times [-\frac{3\varepsilon'}{2}, 0]) - N(K'_0 \cup K'_1),$$

$$H'_2 = N(K'_0 \cup K'_1),$$

where $N(K'_0 \cup K'_1)$ denotes the standard contact neighborhood. Both H'_1 and H'_2 are product disk decomposable. (The product disk decomposability of H'_2 is clear. As for H'_1 , observe that

($T \times [-\frac{\varepsilon'}{2}, 0]$) $- N(K'_1)$ is product disk decomposable.) Hence we may write $H'_1 = S' \times [0, 1]/\sim$, $(x, t) \sim (x, t')$ if $x \in \partial S'$ and $t, t' \in [0, 1]$. Here $\partial S' \times [0, 1]/\sim$ is the dividing set of $\partial H'_1$, and $R_+(\Gamma') \subset S' \times \{1\}$. Let $P' = S' - \overline{R_+(\Gamma')}$. Similarly we can write $H'_2 = S'_{(2)} \times [0, 1]/\sim$. If $-\Gamma'_{-3\varepsilon'/2}$ is the dividing set of $T_{-3\varepsilon'/2}$, with the outward orientation induced from $T \times [-\frac{3\varepsilon'}{2}, 0]$, then let $P'_{(2)} = S'_{(2)} - \overline{R_-(-\Gamma'_{-3\varepsilon'/2})}$. Observe that $P' \times \{1\}$ is identified with $P'_{(2)} \times \{0\}$; let $\psi : P' \xrightarrow{\sim} P'_{(2)}$ be the corresponding identification map. Also let $h' : Q' \rightarrow S'$ be the monodromy map for the partially defined open book, where the domain of definition Q' is a subset of P' and contains arcs that correspond to compressing disks of $N(K'_1)$.

Next let $\{a'_1, \dots, a'_k\}$ be a maximal set of properly embedded arcs on S' such that the corresponding $\alpha'_i = \partial(a'_i \times [0, 1])$, $i = 1, \dots, k$, on $\partial H'_1$ form a maximal collection of curves which can be extended to a full α' set. (We will abuse notation and call such a maximal collection of arcs a *basis* for $(S', R_+(\Gamma'))$). Let b'_i be the usual pushoff of a'_i , and define $\beta'_i = \partial(\psi(b'_i) \times [0, 1])$, $i = 1, \dots, k$, on $\partial H'_2$.

Lemma 4.1. *$\{\alpha'_1, \dots, \alpha'_k\}$ and $\{\beta'_1, \dots, \beta'_k\}$ can be completed to full α' and β' sets which are weakly admissible.*

Proof. The decomposition $T \times [-\frac{3\varepsilon'}{2}, 0] = H'_1 \cup H'_2$ can be extended to a decomposition of M' into two handlebodies. To accomplish this, let K'_2 be the graph defined above, and choose K'_3 to be the graph such that $N(K'_2) \cup N(K'_3)$ is a decomposition of $M' - (T \times [-\frac{3\varepsilon'}{2}, 0])$ into two handlebodies. Then M' is the union of the two handlebodies $H'_1 \cup N(K'_3) = M' - N(K')$ and $H'_2 \cup N(K'_2) = N(K')$. The collection $\{\alpha'_1, \dots, \alpha'_k\}$ can be completed to a full α' set by adding $\alpha'_{k+1}, \dots, \alpha'_{k+l}$ which are meridians of $N(K'_3)$. On the other hand, $\{\beta'_1, \dots, \beta'_k\}$ can be completed by adding meridians of $N(K'_2)$, in addition to $\partial(c'_i \times [0, 1]) \subset \partial H'_2$, where c'_i are properly embedded arcs of $R_-(-\Gamma'_{-3\varepsilon'/2})$. (Add enough compressing disks of $N(K'_2)$ so that $H'_2 \cup N(K'_2)$ compresses to H'_2 . Then add enough arcs c_i so that $S'_{(2)} - \cup_i c'_i - \cup_i \psi(b'_i)$ deformation retracts to the “ends” of $S'_{(2)}$, namely the arcs of intersection with $T_{0,-}$.)

We now prove that the above extension can be made weakly admissible, without modifying α'_i and β'_i , $i = 1, \dots, k$. If a periodic domain uses any α'_i or β'_i with $1 \leq i \leq k$, then the position of $R_+(\Gamma')$ and the relative positions of a'_i and b'_i imply that the periodic domain has both positive and negative signs. Hence assume that we are not using α'_i or β'_i with $1 \leq i \leq k$. It is easy to find disjoint closed curves $\gamma'_{k+1}, \dots, \gamma'_{k+l}$ which are duals of $\alpha'_{k+1}, \dots, \alpha'_{k+l}$, i.e., γ'_i and α'_j have geometric intersection number δ_{ij} , and which do not enter $\partial H'_1$. (Hence the γ'_i do not intersect α'_j with $j = 1, \dots, k$.) If we wind the α'_i , $i = k+1, \dots, k+l$, about the curves γ'_i as in [OS1, Section 5], then the result will be weakly admissible. \square

Remark. An alternate way of thinking of the contact compatibility with respect to ζ near $\partial M'$ is as follows: Start with any Heegaard decomposition $(\Sigma', \beta', \alpha')$ for $(-M', -\Gamma')$. Take $T \times [0, 1]$ with the invariant contact structure ζ , and form a partial open book decomposition for $(T \times [0, 1], \zeta)$ by choosing a Legendrian skeleton consisting of sufficiently many arcs of type $\{p\} \times [0, 1]$, where $p \in \Gamma'$. Let Σ'_ζ be the corresponding contact-compatible Heegaard surface. Then the Heegaard surface for $(-M', -\Gamma')$ which is contact-compatible with respect to ζ near $\partial M'$ is obtained from Σ' by attaching two Heegaard surfaces of type Σ'_ζ , one for $T \times [-\frac{3\varepsilon'}{2}, -\frac{\varepsilon'}{2}]$ and another for $T \times [-\frac{\varepsilon'}{2}, 0]$.

(Note that the choice of arcs of type $\{p\} \times [0, 1]$ for the two Σ'_ζ 's may be different.) In other words, we are gluing two copies of $(T \times [0, 1], \zeta)$ to $(-M', -\Gamma')$.

Remark. Another approach is to restrict attention to the class of contact-compatible Heegaard splittings for an arbitrarily chosen, tight or overtwisted, contact (M', Γ', ξ') compatible with the dividing set. Suppose we show that the definition of Φ_ξ depends only on the partial open book (S', h') for (M', Γ', ξ') , up to positive and negative stabilizations. By the result of [GG], two open books become isotopic after a sequence of positive and negative stabilizations, provided they correspond to homologous contact structures. This would show that Φ_ξ is only dependent on the homology class of ξ' . However, we would still need to remove the dependence on the homology class.

Step 2: Extension of the Heegaard splitting to $(-M, -\Gamma)$. We extend the Heegaard splitting $(\Sigma', \beta', \alpha')$ constructed in Step 1 to a Heegaard splitting (Σ, β, α) for $(-M, -\Gamma)$ which is *contact-compatible with respect to $\xi \cup \zeta$* . (Again, we take the construction below as the definition of contact-compatibility.)

Let $\varepsilon'' > 0$. Then we write $M = M' \cup N \cup M''$, where $N = N_{\varepsilon''} = T \times [0, \varepsilon'']$ and $M'' = M''_{\varepsilon''} = M - \text{int}(M' \cup N)$.

The contact manifold $(M'', \xi|_{M''})$ admits a Legendrian graph K'' with endpoints on $\Gamma_{\partial M''}$ and a decomposition into $N(K'')$ and $M'' - N(K'')$, according to [HKM3, Theorem 1.1]. Assume that every connected component of K'' intersects Γ at least twice. (This will be useful in Lemma 5.1.) Similarly, $(N, \xi|_N)$ admits a Legendrian graph K''' consisting of Legendrian segments $\{q\} \times [0, \varepsilon'']$, where there is at least one q for each component of the dividing set of T_0 . We also assume that the endpoints of K' , K'' , and K''' do not intersect.

We then decompose M into $H_1 = (M' - N(K')) \cup N(K''') \cup (M'' - N(K''))$ and $H_2 = N(K') \cup (N - N(K''')) \cup N(K'')$, respectively. Since $N - N(K''')$ and $N(K'')$ are product disk decomposable with respect to ξ , their union is also product disk decomposable. Hence, we have:

- H_2 is a neighborhood of a graph K ,
- the restriction of K to $M'' \cup (T \times [-\varepsilon', \varepsilon''])$ is Legendrian, and
- restricted to $M'' \cup (T \times [-\frac{3\varepsilon'}{2}, \varepsilon''])$, H_2 is a standard contact neighborhood of $K \cap (M'' \cup (T \times [-\varepsilon', \varepsilon'']))$.

Similarly, $((T \times [-\frac{3\varepsilon'}{2}, 0]) - N(K'_0 \cup K'_1)) \cup N(K''') \cup (M'' - N(K''))$ is product disk decomposable with respect to $\xi \cup \zeta$. Therefore, H_1 extends $H'_1 \cup N(K'_3) = (S' \times [0, 1] / \sim) \cup N(K'_3)$ so that $H_1 = (S \times [0, 1] / \sim) \cup N(K'_3)$, where $(x, t) \sim (x, t')$ if $x \in \partial S$ and $t, t' \in [0, 1]$. Here, $R_+(\Gamma) \subset S \times \{1\}$ and S' is a subsurface of S .

Therefore, we may extend $(\Sigma', \beta', \alpha')$ to (Σ, β, α) as follows: Consider a collection of arcs a''_1, \dots, a''_m which form a basis for $(S - P', R_+(\Gamma))$. Then let $\alpha''_i = \partial(a''_i \times [0, 1])$, and β''_i be the corresponding closed curves derived from the pushoffs b''_i of a''_i . The monodromy h for b''_i can be computed from the partial open book decomposition on $M'' \cup (T \times [-\frac{3\varepsilon'}{2}, \varepsilon''])$. Then $\alpha = \alpha' \cup \alpha''$ and $\beta = \beta' \cup \beta''$, where α'' (resp. β'') is the collection of the α''_i (resp. β''_i). The contact-compatibility on $\Sigma - \Sigma'$ immediately implies that the extension is weakly admissible.

We are now in a position to define the chain map Φ_ξ . Let $(\Sigma', \beta', \alpha')$ be a Heegaard splitting for $(-M', -\Gamma')$ which is contact-compatible near $\partial M'$, and let (Σ, β, α) be a contact-compatible extension of $(\Sigma', \beta', \alpha')$ to $(-M, -\Gamma)$. Now let x''_i be the preferred intersection point (i.e., the only

one on $S \times \{1\}$) between α''_i and β''_i , and denote their collection by \mathbf{x}'' . Given $\mathbf{y} \in CF(\Sigma', \beta', \alpha')$, we define the map:

$$\begin{aligned} \Phi_\xi : CF(\Sigma', \beta', \alpha') &\rightarrow CF(\Sigma, \beta' \cup \beta'', \alpha' \cup \alpha''), \\ \mathbf{y} &\mapsto (\mathbf{y}, \mathbf{x}''). \end{aligned}$$

The fact that Φ_ξ is a chain map follows from observing that every nonconstant holomorphic map which emanates from x''_i must nontrivially intersect $R_+(\Gamma)$. Hence \mathbf{x}'' will be used up, and the only holomorphic maps from $(\mathbf{y}, \mathbf{x}'')$ to $(\mathbf{y}', \mathbf{x}'')$ are holomorphic maps from \mathbf{y} to \mathbf{y}' within Σ' . The tuple \mathbf{x}'' will be called *the EH class on $S - P'$* . It is immediate from the definition of Φ_ξ that when (M', Γ', ξ') is contact, $\Phi_\xi(EH(M', \Gamma', \xi')) = EH(M, \Gamma, \xi' \cup \xi)$.

Remark. Observe that the set $\{a''_1, \dots, a''_m\}$ contains arcs of $R_+(\Gamma') \subset S'$. This is one of the reasons Σ' must be contact-compatible near $\partial M'$.

5. NATURALITY OF Φ_ξ

In this section we prove that Φ_ξ does not depend on the choices made in Section 4. The proofs are similar to the proofs of well-definition of the *EH* class in [HKM2, HKM3], and we will only highlight the differences. The proof of naturality under isotopy is identical to the proof of [HKM2, Lemma 3.3], and will be omitted.

5.1. Handlesliding. Consider the Heegaard surface Σ and two sets of compressing disks (β, α) , $(\bar{\beta}, \bar{\alpha})$ which are contact-compatible with respect to $\xi \cup \zeta$. In particular, (β'', α'') and $(\bar{\beta}'', \bar{\alpha}'')$ correspond to bases $\{a''_1, \dots, a''_m\}$ and $\{\bar{a}''_1, \dots, \bar{a}''_m\}$ for $(S - P', R_+(\Gamma))$.

There are two types of operations to consider:

- (A) Arc slides in the $S - P'$ region, while fixing α' and β' .
- (B) Handleslides within Σ' , while preserving the contact-compatible α'' and β'' .

Lemma 5.1. *Suppose the closure of each component of $S - P' - \overline{R_+(\Gamma)}$ intersects Γ along at least two arcs. Then one can take (β, α) to $(\bar{\beta}, \bar{\alpha})$ through a sequence of moves of type (A) or (B).*

The required connectivity of $S - P' - \overline{R_+(\Gamma)}$ was already incorporated in the definition in Section 4.

Proof. According to [HKM2, Lemma 3.3], any basis $\{a''_i\}_{i=1}^m$ for $(S - P', R_+(\Gamma))$ can be taken to any other basis $\{\bar{a}''_i\}_{i=1}^m$ for $(S - P', R_+(\Gamma))$ through a sequence of arc slides within $S - P' - \overline{R_+(\Gamma)}$, assuming sufficient connectivity of $S - P' - \overline{R_+(\Gamma)}$. We must, however, not forget Σ' . If Σ' is taken into account, the situation given in Figure 2 must be dealt with: Locally P' is attached to $S - P'$ along an arc c' (in the diagram, we have pushed c' into P'), and we would like to arc slide a''_i over c' to obtain \bar{a}''_i . However, this c' may not be an a'_i . If this is the case, we must perform a sequence of handleslides on α' and β' first (while fixing α'' and β''), so that $\alpha'_1 = \partial(c' \times [0, 1])$ and $\beta'_1 = (c' \times \{1\}) \cup (h(c') \times \{0\})$. This is possible since we required at least two arcs $\{p\} \times [-\varepsilon', 0]$ in the definition of K'_1 . Then we may arc slide a''_i over c' . \square

We now discuss naturality under the moves (A) and (B).

(A). Recall that an arc slide corresponds to a sequence of two handleslides by [HKM2]. For each handleslide of an arc slide in the $S - P'$ region, the “tensoring with Θ ” map Ψ sends the *EH* class \mathbf{x}'' on $S - P' - \overline{R_+(\Gamma)}$ to the *EH* class on $S - P' - \overline{R_+(\Gamma)}$, also called \mathbf{x}'' by abuse of notation.

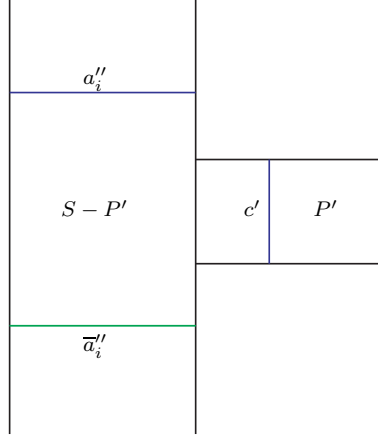


FIGURE 2.

Since the α'' and β'' are used up, the restriction of Ψ to the remaining r -tuple $\mathbf{y} \in CF(\Sigma', \beta', \alpha')$ is the natural “tensoring with Θ ” map Ψ' from $CF(\Sigma', \beta', \alpha')$ to $CF(\overline{\Sigma}', \overline{\beta}', \overline{\alpha}')$. Therefore,

$$\Psi(\mathbf{y}, \mathbf{x}'') = (\Psi'(\mathbf{y}), \mathbf{x}'').$$

The proof is identical to the proof of [HKM2, Lemma 5.2].

(B). The “tensoring with Θ ” operation for a handleslide in the Σ' region clearly sends \mathbf{x}'' to \mathbf{x}'' as well. Therefore we have:

$$\Psi(\mathbf{y}, \mathbf{x}'') = (\Psi'(\mathbf{y}), \mathbf{x}'').$$

5.2. Stabilization. In this subsection we prove naturality under stabilization. For this, we need to prove two things: (A) naturality under stabilizations (contact or otherwise) inside M' , and (B) naturality under positive (contact) stabilizations inside $M - M'$.

Let A be a surface with nonempty boundary and $B \subset A$ be a subsurface. Let c be a properly embedded arc in A ; after isotopy rel boundary, we assume c intersects ∂B transversely and efficiently. Then we define the *complexity of c with respect to (A, B)* as the number of subarcs of c which are contained in B and have both endpoints on the common boundary of $A - B$ and B .

Given two Heegaard splittings $(\Sigma', \beta', \alpha')$ and $(\overline{\Sigma}', \overline{\beta}', \overline{\alpha}')$ for $(-M', -\Gamma')$ which are contact compatible with respect to ζ near $\partial M'$ (i.e., of the type constructed in Step 1 of Section 4) and their extensions (Σ, β, α) and $(\overline{\Sigma}, \overline{\beta}, \overline{\alpha})$ to $(-M, -\Gamma)$ which are contact compatible with respect to $\xi \cup \zeta$ (i.e., of the type constructed in Step 2 of Section 4), we first find a common stabilization $(\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha})$, which is also contact compatible with respect to $\xi \cup \zeta$. If we place a line (resp. tilde) over a symbol, then it stands for the corresponding object for $(\overline{\Sigma}, \overline{\beta}, \overline{\alpha})$ (resp. $(\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha})$), e.g., \overline{K}' is K' for $(\overline{\Sigma}, \overline{\beta}, \overline{\alpha})$. (The exception is $\overline{R}_+(\Gamma')$, which refers to the closure of $R_+(\Gamma')$.)

(A) We will first discuss the subdivision on $(-M', -\Gamma')$. Take $\tilde{\varepsilon}' > 0$ so that $\tilde{\varepsilon}' \ll \varepsilon', \tilde{\varepsilon}'$. Given the Legendrian portion $L'_0 = K'_0 \cup K'_1$ of the 1-skeleton K' , we successively attach Legendrian arcs c'_i to L'_i to obtain L'_{i+1} in the following order:

- (α) First attach arcs to construct the Legendrian 1-skeleton of a sufficiently fine Legendrian cell decomposition of $T_{-\tilde{\varepsilon}'}$, after possibly applying Legendrian realization.
- (β) Then attach Legendrian arcs of the type $\{p\} \times [-\tilde{\varepsilon}', 0]$ with $p \in \Gamma'$.

The arcs are attached so that in the end we obtain a Legendrian graph containing $\tilde{L}'_0 = \tilde{K}'_0 \cup \tilde{K}'_1$, where \tilde{K}'_0 is a Legendrian skeleton of $T_{-\varepsilon'}$ and \tilde{K}'_1 is the union of arcs of type $\{p\} \times [-\varepsilon', 0]$, and so that the restrictions of L'_0 and \bar{L}'_0 to $T \times [-\varepsilon', 0]$ are subsets of \tilde{L}'_0 . If we start with $\bar{L}'_0 = \bar{K}'_0 \cup \bar{K}'_1$ instead, then there is a sequence \bar{L}'_i which eventually yields a Legendrian graph containing \tilde{L}'_0 . The stabilizations of contact type will be treated in (A₁). Next extend \tilde{L}'_0 to a common refinement \tilde{K}' of K' and \bar{K}' by subdividing on $M' - (T \times [-\varepsilon', 0])$. These stabilizations will be treated in (A₂).

(A₁) The above attachments of Legendrian arcs are done in the same way as in [HKM3, Theorem 1.2] and in particular [HKM3, Figures 1, 3, and 4].

The attachment of arcs $c'_i \subset T_{-\varepsilon'}$ of type (α) can be decomposed into three stages (α_1) , (α_2) and (α_3) . Figure 3 depicts an arc of type (α_1) . An arc c'_i of type (α_1) connects between $\{p\} \times [-\varepsilon', 0]$ and $\{q\} \times [-\varepsilon', 0]$, where $p, q \in \Gamma'$. After attaching all the arcs of type (α_1) , we attach the arcs of type (α_2) , depicted in Figure 4. Here, the arc c'_i connects between two arcs of type (α_1) and does not cross the dividing set of $T_{-\varepsilon'}$. Finally, an arc of type (α_3) is an arc that intersects the dividing set of $T_{-\varepsilon'}$ exactly once, and in its interior. Arcs of type (α_1) , (α_2) , and (α_3) are sufficient to construct the Legendrian skeleton of $T_{-\varepsilon'}$. Figure 5 depicts an arc attachment of type (β) .

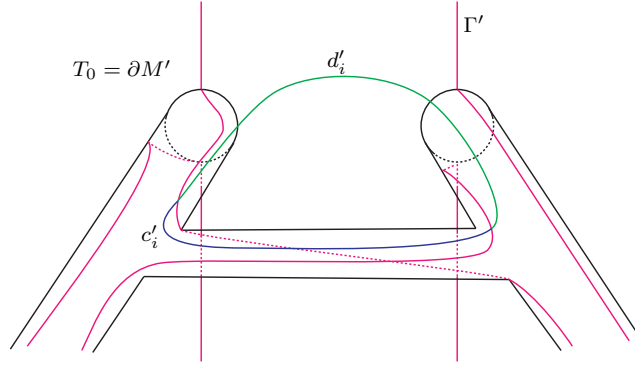
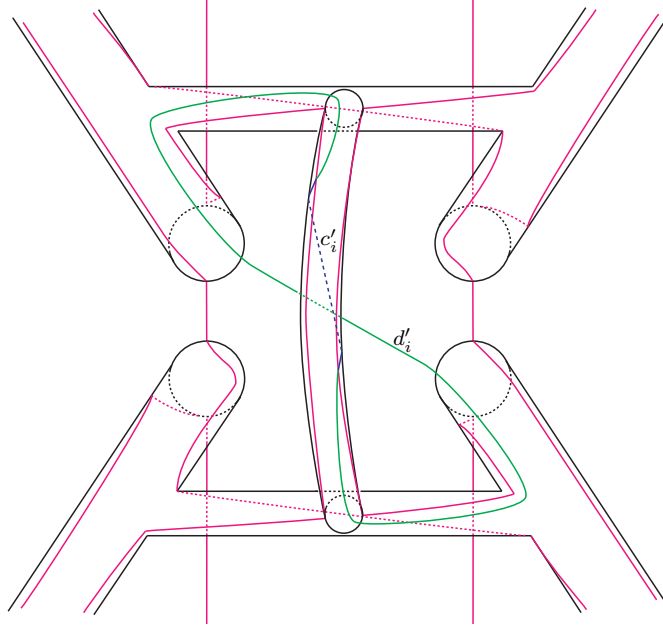
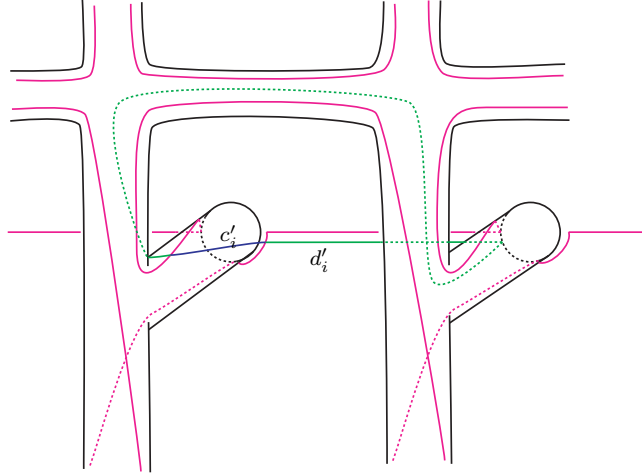


FIGURE 3. Arc of type (α_1) . The surface in the back is $T_0 = \partial M'$, whose orientation as the boundary of M' points into the page. The cylinders on the left and right are thickenings of arcs $\{p\} \times [-\varepsilon', 0]$ and $\{q\} \times [-\varepsilon', 0]$ of K'_1 , and the horizontal cylinder is a thickening of c'_i . The blue arc is c'_i and the green arc is its isotopic copy d'_i .

In particular, we observe that the following holds:

- Each endpoint of c'_i lies on $\Gamma_{\partial(M' - N(L'_i))}$, and $\text{int}(c'_i) \subset \text{int}(M' - N(L'_i))$.
- $N(L'_{i+1}) = N(c'_i) \cup N(L'_i)$, and L'_{i+1} is a Legendrian graph so that $N(L'_{i+1})$ is its standard neighborhood.
- There is a Legendrian arc d'_i on $\partial(M' - N(L'_i))$ with the same endpoints as c'_i , after possible application of the Legendrian realization principle. The arc d'_i intersects $\Gamma_{\partial(M' - N(L'_i))}$ only at its endpoints.
- The Legendrian knot $\gamma'_i = c'_i \cup d'_i$ bounds a disk in $M' - N(L'_i)$ and has $tb(\gamma'_i) = -1$ with respect to this disk. This implies that c'_i and d'_i are isotopic relative to their endpoints inside the closure of $M' - N(L'_i)$.

FIGURE 4. Arc of type (α_2) .FIGURE 5. Arc of type (β) .

For simplicity, consider the situation of attaching a single arc c'_0 to L'_0 to obtain \tilde{L}'_0 . Consider the (very) partial open book decomposition on $T \times [-\frac{3\varepsilon'}{2}, 0]$, corresponding to the decomposition into $H'_2 = N(K'_0 \cup K'_1)$ and $H'_1 = (T \times [-\frac{3\varepsilon'}{2}, 0]) - N(K'_0 \cup K'_1) = S' \times [0, 1] / \sim$. The monodromy map is $h' : Q' \rightarrow S'$ as before. The arc c'_0 can be viewed as a Legendrian arc on $S' \times \{\frac{1}{2}\}$ with endpoints on $\partial Q' \times \{\frac{1}{2}\}$. Hence, removing a neighborhood of c'_0 from H'_1 and adding it to H'_2 is equivalent to the following positive (contact) stabilization: Let e'_0 be the Legendrian arc on $S' = S' \times \{1\}$ which is Legendrian isotopic to c'_0 rel endpoints, via an isotopy inside H'_1 . Add a 1-handle to S' along the endpoints of e'_0 to obtain \tilde{S}' , and complete e'_0 to a closed curve γ_0 on \tilde{S}' by attaching the

core of the 1-handle. Then the stabilization is the data $(\tilde{S}', R_+(\Gamma'), \tilde{Q}', \tilde{h}' = R_{\gamma_0} \circ h')$, where R_{γ_0} is a positive Dehn twist about γ_0 and \tilde{Q}' is the domain of \tilde{h}' . Let (S, h) (resp. (\tilde{S}, \tilde{h})) be the partial open book which extends (S', h') (resp. (\tilde{S}', \tilde{h}')).

Lemma 5.2. *The arc c'_0 can be chosen so that the corresponding $e'_0 \subset S'$ has complexity 0 with respect to (S', P') and complexity at most 1 with respect to $(S', R_+(\Gamma'))$.*

Proof. We treat the (α_1) case, and leave the other cases to the reader. Refer to Figure 3; in the figure replace c'_i, d'_i by c'_0, d'_0 . If d'_0 intersects $R_+(\Gamma')$, then d'_0 , viewed on $S' \times \{1\}$, is the desired isotopic copy e'_0 of c'_0 . It is clear that e'_0 has complexity 0 with respect to (S', P') and complexity 1 with respect to $(S', R_+(\Gamma'))$. On the other hand, if d'_0 intersects $R_-(\Gamma')$, then we need to isotop c'_0 towards $T_{-\varepsilon'/2}$ instead, in order to obtain e'_0 . The procedure is still the same — in Figure 3 assume that the surface in the back is $T_{-\varepsilon'/2}$ (instead of T_0) so that $\partial(M' - (T \times (-\frac{\varepsilon'}{2}, 0]))$ points out of the page. The resulting e'_0 has complexity 0 with respect to both (S', P') and $(S', R_+(\Gamma'))$. \square

In view of Lemma 5.2, there exists a basis $\{a'_1, \dots, a'_k\}$ of $(S', R_+(\Gamma'))$ and an extension to a basis $\{a'_1, \dots, a'_k, a''_1, \dots, a''_m\}$ of $(S, R_+(\Gamma))$, so that e'_0 does not intersect any basis element. Let a'_0 be the cocore of the 1-handle of the stabilization along e'_0 , and let b'_0 be the pushoff of a'_0 . Then let $\alpha'_0 = \partial(a'_0 \times [0, 1])$ and $\beta'_0 = (b'_0 \times \{1\}) \cup (R_{\gamma_0}(b'_0) \times \{0\})$, where both are viewed on $\partial H'_1$. Observe that β'_0 does not intersect any of $\{\alpha'_1, \dots, \alpha'_k, \alpha''_1, \dots, \alpha''_m\}$, where $\alpha'_i = \partial(a'_i \times [0, 1])$ and $\alpha''_i = \partial(a''_i \times [0, 1])$. Since β'_0 also does not intersect the remaining α' -curves $\alpha'_{k+1}, \dots, \alpha'_{k+l}$, the only intersection between β'_0 and some α -curve is the sole intersection with α'_0 .

Let $\Psi : CF(\Sigma, \beta, \alpha) \rightarrow CF(\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha})$ be the composition of (Heegaard decomposition) stabilization and handleslide maps corresponding to the stabilization along e'_0 . We have the following:

Lemma 5.3. *The map*

$$\Psi : CF(\Sigma, \beta, \alpha) \rightarrow CF(\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha})$$

is given by:

$$(\mathbf{y}, \mathbf{x}'') \mapsto (\Psi'(\mathbf{y}), \tilde{\mathbf{x}}''),$$

where $\mathbf{y} \in CF(\Sigma', \beta', \alpha')$, \mathbf{x}'' (resp. $\tilde{\mathbf{x}}''$) is the EH class in the $S - P'$ region (resp. $\tilde{S} - \tilde{P}'$ region) for (Σ, β, α) (resp. $(\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha})$), and Ψ' is the natural map from $(\Sigma', \beta', \alpha')$ to $(\tilde{\Sigma}', \tilde{\beta}', \tilde{\alpha}')$.

Proof. This follows from the technique in [HKM3, Lemma 3.5]. We use the fact that the only intersection between β'_0 and an α -curve is the unique intersection with α'_0 . We decompose the positive stabilization along e'_0 into a trivial stabilization, followed by a sequence of handleslides. (By a *trivial stabilization* we mean the addition of a 1-handle to Σ' , together with curves α'_0 and β'_0 that intersect each other once, say at x'_0 , and no other $\alpha'_i, \beta'_i, i = 1, \dots, s$, and where the regions of $\Sigma' - \cup_{i=0}^s \alpha'_i - \cup_{i=0}^s \beta'_i$ adjacent to x'_0 are path-connected to $R_+(\Gamma')$.) This is done exactly as described in [HKM3, Lemma 3.5]: whenever β'_i (could be β'_i) intersects $e'_0 \times \{0\}$, and $\bar{\beta}'_i$ is the result of applying a positive Dehn twist about $\gamma_0 \times \{0\}$, then $\bar{\beta}'_i$ can be obtained from β'_i by applying a trivial stabilization, followed by handleslides over β'_0 as in [HKM3, Figure 7]. Here, the triple diagrams are weakly admissible for the same reasons as [HKM3, Lemma 3.5].

The slight complication that we need to keep in mind is that the arcs $h(b''_i)$, where b''_i is the usual pushoff of a''_i in $S - P'$, may enter the region S' and intersect e'_0 . If $h(b''_i)$ intersects e'_0 , then the “tensoring with Θ ” map corresponding to handlesliding β''_i over β'_0 sends EH to EH in the

$S - P'$ region and restricts to the natural “tensoring with Θ ” map in the Σ' region. (The proof is the same as that of [HKM3, Lemma 3.5]. Also refer to [HKM3, Figure 8].) On the other hand, if β'_i intersects $e'_0 \times \{0\}$, the $S - P'$ region is unaffected (hence EH is mapped to EH in the $S - P'$ region), and we are doing a standard handleslide map in the Σ' region. \square

(A₂) Next we discuss the effect of a stabilization, in the handlebody sense, in the portion of M' which is not contact-compatible, i.e., away from $T \times [-\tilde{\varepsilon}', 0]$. Assume all the contact stabilizations have already taken place on $T \times [-\tilde{\varepsilon}', 0]$. By abuse of notation, we reset $\varepsilon' = \tilde{\varepsilon}'$ and use the same notation $S', P', Q', h', \Sigma', K', K'_0, K'_1, K'_2, K'_3, H'_1, H'_2$, used in Step 1 of Section 4, for the new (finer) Heegaard decomposition which is contact-compatible on $T \times [-\varepsilon', 0] = T \times [-\tilde{\varepsilon}', 0]$.

Claim. *The stabilization can be decomposed into a trivial stabilization, followed by a sequence of handleslides which avoids $R_-(\Gamma')$.*

Proof. Observe that the arc of stabilization c'_0 is contained in $N(K'_3)$. The meridian of the tubular neighborhood of c'_0 will be called β'_0 and it is not difficult to see that there exists a curve α'_0 which intersects β'_0 once and lies on $\partial N(K'_3) - \partial H'_1$. (Note that β'_0 only intersects α'_0 .) After a sequence of handleslides that takes place away from $\partial H'_1$ (in fact the change takes place in a neighborhood of $\alpha'_0 \cup \beta'_0$), we may assume that α'_0 and β'_0 satisfy the conditions of a trivial stabilization. \square

The claim implies that the handleslides and stabilization do not interact with $h(a''_i)$ (or equivalently with β''_i). Hence the EH class is mapped to the EH class in the $S - P'$ region, and we are doing a standard sequence of handleslide maps plus one stabilization in the Σ' region.

(B) Next we discuss the subdivision on $M - M'$. Take $\tilde{\varepsilon}'' > 0$ so that $\tilde{\varepsilon}'' \ll \varepsilon'', \bar{\varepsilon}''$. Consider $N_{\tilde{\varepsilon}''} = T \times [0, \tilde{\varepsilon}'']$. On $N_{\tilde{\varepsilon}''}$ attach the following arcs in the given order to K''' to obtain a common refinement of the restriction of K''' and \overline{K}''' to $N_{\tilde{\varepsilon}''}$:

- (1) First attach the Legendrian skeleton of a sufficiently fine Legendrian cell decomposition of $T_{\tilde{\varepsilon}''}$, after possibly applying Legendrian realization.
- (2) Then attach Legendrian arcs of type $\{q\} \times [0, \tilde{\varepsilon}'']$ with $q \in \Gamma'$.

Each of the above Legendrian arc attachments leads to a stabilization — however, since the arcs are contained in the complement of $S \times [0, 1]/\sim$, the stabilization is a *precomposition* $h \mapsto h \circ R_\gamma$. More precisely, let c be the Legendrian attaching arc. There is an isotopy of c rel endpoints, inside the complement of $S \times [0, 1]/\sim$, to an arc $e \subset S - P'$, viewed on $S \times \{1\}$, and also to $h(e)$, viewed on $S \times \{0\}$. Observe that $h(e)$ may enter the $R_-(\Gamma')$ region. Add a 1-handle to S along the endpoints of e to obtain \tilde{S} , and complete e to a closed curve γ by attaching the core of the handle.

In the following lemma, we identify $S = S \times \{0\}$ and determine the complexity of the restriction of $h(e)$ to S' and to P' . Observe that $h(e) \cap S' = h(e) \cap \overline{R_-(\Gamma')}$.

Lemma 5.4.

- (1) $h(e)$ has complexity at most one with respect to (S, S') .
- (2) $h(e)$ has complexity at most one with respect to (S, P') .

Proof. We isotop c rel endpoints in two stages: first through the product structure given by the complement of $S \times [0, 1]/\sim$, and then through the product structure given by $S \times [0, 1]/\sim$.

(1) follows from examining the proof of [HKM3, Theorem 1.2] as in Lemma 5.2. The three types of arc attachments are (α_1) , (α_2) , and (α_3) . Consider an arc of type (α_1) , given in Figure 3. In the current case, the surface in the back is still $T_0 = \partial M'$, but the orientation is pointing out of the page; also c'_i and d'_i should be changed to c and d . If the arc d intersects $R_-(\Gamma')$ (where the orientation on $\partial M'$ is the orientation induced from M'), then d is an arc on $S \times \{0\}$, which means that $d = h(e)$. Hence $h(e)$ has complexity 1 with respect to $(S, R_-(\Gamma'))$, and also complexity at most 1 with respect to (S, S') . On the other hand, if d intersects $R_+(\Gamma')$, then $d = e$. Hence $h(e)$ is contained in $(S - S') \times \{0\}$ and has complexity zero with respect to (S, S') . It follows that $h(e)$ also has complexity zero with respect to (S, P') . The arcs of type (α_2) and (α_3) are treated similarly.

(2) follows from considerations similar to [HKM3, Section 5, Example 2]. Suppose $d = h(e)$, i.e., d intersects $R_-(\Gamma')$. (The situation of $d = e$ is easier, and is left to the reader.) Then Figure 6 depicts what happens when we push $h(e)$, viewed as an arc on $S \times \{0\}$, to $S \times \{1\}$. The surface in the front is T_0 and the surface in the back is $T_{-\varepsilon'/2}$. The blue arc $(h(e)|_{S'})_0$ is the isotopic copy of $h(e)|_{S'}$ on T_0 or $S' \times \{0\}$, and the green arc $(h(e)|_{S'})_{-\varepsilon'/2}$ is the copy on $S' \times \{1\}$ which intersects $T_{-\varepsilon'/2}$. We easily see that $h(e)$ has complexity 1 with respect to (S, P') . The arc corresponding to

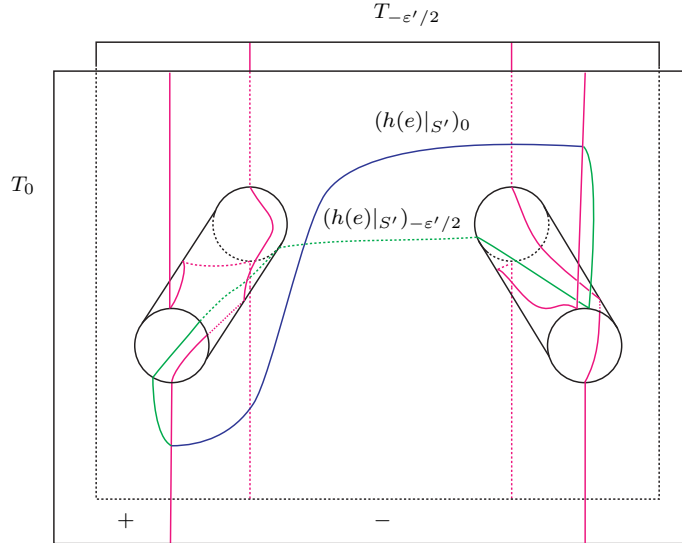


FIGURE 6.

[HKM3, Figure 4] is simpler, and does not enter P' . □

The following lemma follows from Lemma 5.4.

Lemma 5.5. *There exists a basis $\{a'_1, \dots, a'_k\}$ for $(S', R_+(\Gamma'))$ and basis $\{a''_1, \dots, a''_m\}$ for $(S - P', R_+(\Gamma))$ such that the following hold:*

- (1) a''_i does not intersect e for all i ;
- (2) all but one of the a'_i or a''_i are disjoint from $h(e)$;
- (3) one of a'_1 or a''_1 intersects $h(e)$.

The basis $\{a'_1, \dots, a'_k\}$ can be used to construct α', β' for Σ' , and the basis $\{a''_1, \dots, a''_m\}$ gives an extension to α, β on Σ .

Proof. Consider the (α_1) case where d intersects $R_-(\Gamma')$. By (2) of Lemma 5.4, there exists a basis $\{a'_1, \dots, a'_k\}$ for $(S', R_+(\Gamma'))$ so that a'_1 intersects $h(e)$ once, and the remaining $a'_i, i = 2, \dots, k$, do not intersect $h(e)$. Next observe that $e \subset S - P'$ and does not intersect the $R_+(\Gamma)$ region. It is possible to choose a basis $\{a''_1, \dots, a''_m\}$ for $(S - P', R_+(\Gamma))$ which does not intersect e , as well as $h(e)$. The other cases are similar. \square

Let us consider the case where a'_1 intersects $h(e)$. (The other case is similar.) When we stabilize S along e , we add the cocore a''_0 of the 1-handle and obtain the corresponding α''_0 and β''_0 . The only intersection point of α''_0 with any β arc is with β''_0 , which we call x''_0 . Hence we expect the following diagram to commute:

$$\begin{array}{ccc} SFH(\beta', \alpha') & \xrightarrow{\Phi_\xi} & SFH(\beta, \alpha) \\ \Psi \downarrow & & \downarrow \Psi \\ SFH(\beta', \alpha') & \xrightarrow{\Phi_\xi} & SFH(\beta \cup \{\beta''_0\}, \alpha \cup \{\alpha''_0\}) \end{array}$$

However, our stabilization is not a trivial stabilization, as α'_1 intersects β''_0 in one point. Therefore we need to decompose the stabilization into a trivial stabilization, followed by a handleslide. This will be done in a manner similar to [HKM3, Lemma 3.5]. Let γ''_i be pushoffs of α''_i for all i , γ'_j be pushoffs of α'_j for all $j \neq 1$, and γ'_1 be obtained by pushing α'_1 over α''_0 , as depicted in Figure 7. In Figure 7, we place black dots in regions that are path-connected to Γ ; in other words, holomorphic curves are not allowed to enter such regions.

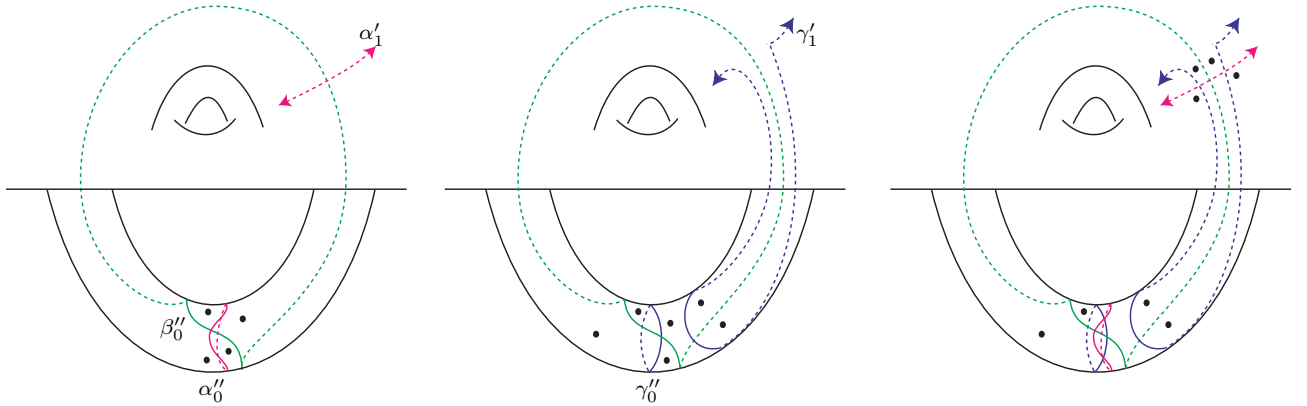


FIGURE 7.

Now consider the following diagram:

$$\begin{array}{ccc}
 SFH(\beta', \gamma') & \xrightarrow{\Phi_\xi} & SFH(\beta, \gamma) \\
 \Psi_1 \downarrow & & \downarrow \Psi_2 \\
 & & SFH(\beta \cup \{\beta_0''\}, \gamma \cup \{\gamma_0''\}) \\
 & & \downarrow \Psi_3 \\
 SFH(\beta', \alpha') & \xrightarrow{\Phi_\xi} & SFH(\beta \cup \{\beta_0''\}, \alpha \cup \{\alpha_0''\})
 \end{array}$$

For the term $SFH(\beta', \gamma')$ in the upper left-hand corner, γ' is the set consisting of all the γ'_i ; for $SFH(\beta, \gamma)$ in the upper right-hand corner, γ is the set consisting of all the γ''_i and γ'_i , with the exception of γ_0'' . This means that (β, γ) is obtained from the middle diagram of Figure 7 by a destabilization; hence γ effectively consists of pushoffs of α . The map Ψ_2 is the map which corresponds to the trivial stabilization, and Ψ_3 is the handleslide map which is the “tensoring with Θ ” map, where Θ is the top generator of $CF(\gamma \cup \{\gamma_0''\}, \alpha \cup \{\alpha_0''\})$. The slightly tricky feature of this diagram is that at $SFH(\beta \cup \{\beta_0''\}, \gamma \cup \{\gamma_0''\})$ we leave the category of diagrams which nicely decompose into the M' part and the $M - M'$ part. The map Ψ_1 is the “tensoring with Θ' ” map, where Θ' is the top generator of $CF(\gamma', \alpha')$. The maps Φ_ξ are the “tensoring with the EH class” maps. By the placement of the dots in the right-hand diagram of Figure 7, it is not difficult to see the following:

Lemma 5.6. *The EH class on $(\beta'' \cup \{\beta_0''\}, \gamma'' \cup \{\gamma_0''\})$ is mapped to the EH class on $(\beta'' \cup \{\beta_0''\}, \alpha'' \cup \{\alpha_0''\})$ via Ψ_3 .*

Proof. The Heegaard triple diagram is weakly admissible for the same reason as Lemma 3.5 of [HKM3], and the details are left to the reader. In the right-hand diagram of Figure 7, consider the largest closed connected component R which is bounded by the $\alpha \cup \{\alpha_0''\}, \beta \cup \{\beta_0''\}, \gamma \cup \{\gamma_0''\}$ curves, does not contain a dot (i.e., does not intersect Γ), and contains the unique intersection point of β_0'' and γ_0'' . The set R is an annulus which is bounded by α_0'' and γ_0'' on one side, and by α'_1 and γ'_1 on the other. There are two points of Θ in R , but only one intersection point of $\beta \cup \{\beta_0''\}$ and $\gamma \cup \{\gamma_0''\}$. Hence one of the Θ points cannot be used towards R , namely the intersection point between α'_1 and γ'_1 . This allows us to “erase” the boundary component of R consisting of α'_1 and γ'_1 , and conclude that $\alpha_0'' \cap \gamma_0''$ is mapped to $\alpha_0'' \cap \beta_0''$. The rest of the tuples of the EH class are straightforward. \square

Once the EH portion is used up by Lemma 5.6, Ψ_3 restricts to Ψ_1 on the rest of the tuples, i.e., those that lie on Σ' . The commutativity of the above diagram follows.

Now, inside $M''_{\tilde{\Sigma}'} = M - \text{int}(M' \cup N_{\tilde{\Sigma}'}')$, we attach Legendrian arcs to the Legendrian graph which plays the role of K'' so that we have a common refinement of K'' and \overline{K}'' . An arc attachment in this region corresponds to a straightforward stabilization along c which lies in $S - S'$. The map

on Floer homology induced by such a stabilization clearly sends EH to EH and has a natural restriction to the Σ' region.

6. PROPERTIES OF THE GLUING MAP

In this section we collect some standard properties of the gluing map.

Theorem 6.1 (Identity). *Let (M, Γ) be a sutured manifold and ξ be a $[0, 1]$ -invariant contact structure on $\partial M \times [0, 1]$ with dividing set $\Gamma \times \{t\}$ on $\partial M \times \{t\}$. The gluing map*

$$\Phi_\xi : SFH(-M, -\Gamma) \rightarrow SFH(-M, -\Gamma),$$

obtained by attaching $(\partial M \times [0, 1], \xi)$ onto (M, Γ) along $\partial M \times \{0\}$, is the identity map (up to an overall \pm sign if over \mathbb{Z}).

The proof of Theorem 6.1 will be given in Subsection 6.1, after some preliminaries.

Proposition 6.2 (Composition). *Consider the inclusions $(M_1, \Gamma_1) \subset (M_2, \Gamma_2) \subset (M_3, \Gamma_3)$ of sutured manifolds, and let ξ_{12} be a contact structure on $M_2 - \text{int}(M_1)$ which has convex boundary and dividing sets Γ_i on ∂M_i , $i = 1, 2$. Similarly define ξ_{23} . If*

$$\Phi_{12} : SFH(-M_1, -\Gamma_1) \rightarrow SFH(-M_2, -\Gamma_2),$$

$$\Phi_{23} : SFH(-M_2, -\Gamma_2) \rightarrow SFH(-M_3, -\Gamma_3),$$

$$\Phi_{13} : SFH(-M_1, -\Gamma_1) \rightarrow SFH(-M_3, -\Gamma_3),$$

are natural maps induced by ξ_{12} , ξ_{23} , and $\xi_{12} \cup \xi_{23}$, respectively, then $\Phi_{23} \circ \Phi_{12} = \Phi_{13}$, (up to an overall \pm sign if over \mathbb{Z}).

Proof. This is immediate, once we unwind the definitions. Let $(S_1, R_+(\Gamma_1), h_1)$ be a partial open book decomposition for (M_1, Γ_1, ξ_1) . Here ξ_1 is arbitrary and may be tight or overtwisted. Let $(\Sigma_1, \beta_1, \alpha_1)$ be the corresponding contact-compatible Heegaard splitting. We assume that the partial open book for ξ_1 is sufficiently fine and the Heegaard splitting is of the type given in Step 1 of Section 4. Extend $(S_1, R_+(\Gamma_1), h_1)$ to $(S_2, R_+(\Gamma_2), h_2)$ via ξ_{12} (of the type given in Step 2 of Section 4), and let \mathbf{x}_{12} be the EH class for the arcs which complete a basis for $(S_1, R_+(\Gamma_1), h_1)$ to a basis for $(S_2, R_+(\Gamma_2), h_2)$. Similarly define \mathbf{x}_{23} . Then the chain map Φ_{12} maps:

$$\mathbf{y} \mapsto (\mathbf{y}, \mathbf{x}_{12}),$$

and Φ_{23} maps:

$$(\mathbf{y}, \mathbf{x}_{12}) \mapsto (\mathbf{y}, \mathbf{x}_{12}, \mathbf{x}_{23}).$$

This is the same as $\Phi_{13}(\mathbf{y})$, since $(\mathbf{x}_{12}, \mathbf{x}_{23})$ is the EH class for the arcs which complete a basis for $(S_1, R_+(\Gamma_1), h_1)$ to a basis for $(S_3, R_+(\Gamma_3), h_3)$. Moreover the extension is of the type given in Step 2 of Section 4. \square

Proposition 6.3 (Associativity). *Let (M_1, Γ_1) , (M_2, Γ_2) , and (M_3, Γ_3) be pairwise disjoint sutured submanifolds of (M, Γ) . Let ξ be a contact structure defined on $M - \text{int}(M_1 \cup M_2 \cup M_3)$ which has convex boundary and dividing sets Γ on ∂M and Γ_i on ∂M_i . Let (M_{12}, Γ_{12}) be a sutured submanifold of (M, Γ) which is disjoint from M_3 , contains M_1 and M_2 , and has dividing set Γ_{12} on ∂M_{12} with respect to ξ . Similarly define (M_{23}, Γ_{23}) . Then the maps*

$$(6.0.1) \quad SFH(-M_1, -\Gamma_1) \otimes SFH(-M_2, -\Gamma_2) \otimes SFH(-M_3, -\Gamma_3)$$

$$\xrightarrow{\Phi_{\xi|M_{12}-M_1-M_2} \otimes id} SFH(-M_{12}, -\Gamma_{12}) \otimes SFH(-M_3, -\Gamma_3) \xrightarrow{\Phi_{\xi|M-M_{12}-M_3}} SFH(-M, -\Gamma)$$

and

$$(6.0.2) \quad SFH(-M_1, -\Gamma_1) \otimes SFH(-M_2, -\Gamma_2) \otimes SFH(-M_3, -\Gamma_3)$$

$$\xrightarrow{id \otimes \Phi_{\xi|M_{23}-M_2-M_3}} SFH(-M_1, -\Gamma_1) \otimes SFH(-M_{23}, -\Gamma_{23}) \xrightarrow{\Phi_{\xi|M-M_1-M_{23}}} SFH(-M, -\Gamma)$$

are identical (up to an overall \pm sign if over \mathbb{Z}).

Proof. Let $(S_i, R_+(\Gamma_i), h_i)$ be a partial open book decomposition for (M_i, Γ_i, ξ_i) , $i = 1, 2, 3$, where ξ_i is arbitrary. Let $(\Sigma_i, \beta_i, \alpha_i)$ be the corresponding contact-compatible Heegaard splitting. To define the chain map $\Phi_{12} = \Phi_{\xi|M_{12}-M_1-M_2}$, we extend $(S_i, R_+(\Gamma_i), h_i)$ to a partial open book decomposition $(S_{12}, R_+(\Gamma_{12}), h_{12})$ for $(M_{12}, \Gamma_{12}, \xi|_{M_{12}-M_1-M_2} \cup \xi_1 \cup \xi_2)$. Then

$$\Phi_{12} : (\mathbf{y}_1, \mathbf{y}_2) \mapsto (\mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_{12}),$$

where \mathbf{x}_{12} is the EH class for the arcs which complete a basis for

$$(S_1, R_+(\Gamma_1), h_1) \cup (S_2, R_+(\Gamma_2), h_2)$$

to a basis for $(S_{12}, R_+(\Gamma_{12}), h_{12})$. Next we complete a basis for

$$(S_{12}, R_+(\Gamma_{12}), h_{12}) \cup (S_3, R_+(\Gamma_3), h_3)$$

to a basis for the open book $(S_{123}, R_+(\Gamma_{123}), h_{123})$ corresponding to $(M, \xi \cup \xi_1 \cup \xi_2 \cup \xi_3)$, and let $\mathbf{x}_{(12)3}$ be the EH class for the completing arcs. Hence, $\Phi_{(12)3} = \Phi_{\xi|M-M_{12}-M_3}$ maps:

$$(\mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_{12}) \otimes \mathbf{y}_3 \mapsto (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{x}_{12}, \mathbf{x}_{(12)3}).$$

Similarly, $\Phi_{1(23)} \circ (id \otimes \Phi_{23})$ sends

$$(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) \mapsto (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{x}_{23}, \mathbf{x}_{1(23)}).$$

By applying a sequence of positive stabilization and basis change moves in the $M - \text{int}(M_1 \cup M_2 \cup M_3)$ region, as proven in Section 5, we see that $(\mathbf{x}_{12}, \mathbf{x}_{(12)3})$ is taken to $(\mathbf{x}_{23}, \mathbf{x}_{1(23)})$. \square

Proposition 6.4. *Let (M', Γ') be obtained from (M, Γ) by decomposing along a properly embedded surface T with ∂ -parallel dividing set Γ_T . The inclusion/direct summand map*

$$SFH(-M', -\Gamma') \rightarrow SFH(-M, -\Gamma)$$

given in [HKM3, Section 6] is the same as the gluing map of Theorem 1.3.

Proposition 6.4 can be proved using techniques as that are similar to the proof of Theorem 6.1 below, and is left to the reader.

6.1. Proof of Theorem 6.1. In this subsection we prove Theorem 6.1.

6.1.1. *Attaching a trivial bypass.* Let $(S', R_+(\Gamma'), h')$ be a partial open book decomposition for the triple (M', Γ', ξ') , where ξ' is any contact structure. We determine the effect of attaching a trivial bypass on the partial open book $(S', R_+(\Gamma'), h')$. Let (M, Γ) be the result of attaching a bypass D to (M', Γ') along a trivial arc of attachment c , and thickening. (Of course (M, Γ) and (M', Γ') are isotopic, but we keep the distinction.) The boundary ∂D decomposes into two arcs which intersect at their common endpoints: the arc of attachment $c \subset \partial M'$ and the bypass arc d . As described in [HKM3, Section 5, Example 5], attaching a neighborhood $N(D)$ of D is equivalent to attaching a tubular neighborhood of d (a 1-handle), followed by a neighborhood $D_0 \times [0, 1]$ of a disk $D_0 \subset D$ which is a slight retraction of D (a 2-handle). Now, let K' be the Legendrian graph in (M', Γ') with endpoints on Γ' , which gives rise to the partial open book decomposition $(S', R_+(\Gamma'), h')$. Then the Legendrian graph K for $(S, R_+(\Gamma), h)$ is obtained from K' by taking the union with a Legendrian arc $\{pt\} \times [0, 1] \subset D_0 \times [0, 1]$, which is the cocore of the 2-handle. The complement of $N(K)$ in M is product disk decomposable. This decomposition gives rise to an extension $(S, R_+(\Gamma), h)$ of $(S', R_+(\Gamma'), h')$ to (M, Γ) , obtained by attaching a 1-handle to S' . Let a_0 be the cocore of the 1-handle. The monodromy h' on the S' -portion remains unchanged.

We now apply the calculations done in [HKM3, Section 5, Example 5] to obtain a description of $(S, R_+(\Gamma), h)$. There are two cases of trivial bypasses: c cuts off a half-disk D_1 of $\partial M' - \Gamma'$ which is either in $R_+(\Gamma')$ or in $R_-(\Gamma')$. (If c cuts off two half-disks D_1, D_2 and $\partial D_1, \partial D_2$ intersect along an arc of Γ' , then we take D_1 to be the “smaller” half-disk, i.e., $\partial D_1 \cap \Gamma' \subset \partial D_2 \cap \Gamma'$.) The two cases will be called the R_+ and R_- cases, respectively. See Figure 8 for the determination of the monodromy corresponding to the portion that is attached.

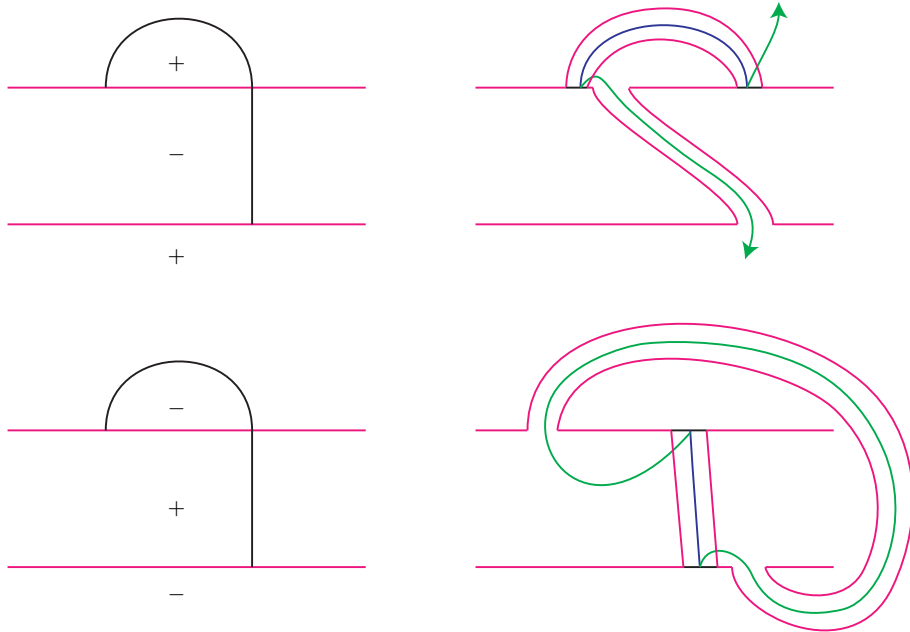


FIGURE 8. The top row is the R_+ case and the bottom row is the R_- case. The diagrams on the right-hand side depict the 1-handle attached to S' to obtain S . The blue arc a_0 completes a basis for $(S', R_+(\Gamma'), h')$ to a basis $(S, R_+(\Gamma), h)$, and the green arc is its image $h(a_0)$.

6.1.2. Effect of a trivial bypass attachment on sutured Floer homology. Let $(\Sigma', \beta', \alpha')$ be the contact-compatible Heegaard splitting for a basis $\{a'_1, \dots, a'_k\}$ for $(S', R_+(\Gamma'), h')$ and $(\Sigma, \beta = \beta' \cup \{\beta_0\}, \alpha = \alpha' \cup \{\alpha_0\})$ be its extension with respect to $\{a'_1, \dots, a'_k, a_0\}$ for $(S, R_+(\Gamma), h)$. Here $\alpha_0 = \partial(a_0 \times [0, 1])$ and $\beta_0 = (b_0 \times \{1\}) \cup (h(b_0) \times \{1\})$, where b_0 is the usual pushoff of a_0 . Let x_0 be the EH class corresponding to a_0 .

Let c be the trivial arc of attachment along $\partial M'$ and let $D_1 \subset \partial M'$ be the half-disk cobounded by a subarc of c and an arc of Γ' , as described above. Assume, without loss of generality, that no endpoint of K' lies on ∂D_1 . If $D_1 \subset R_+(\Gamma')$, then the only intersection of α_0 with any β_i is x_0 . On the other hand, if $D_1 \subset R_-(\Gamma')$, then the only intersection of β_0 with any α_i is x_0 . Therefore, for both R_+ and R_- , the inclusion map

$$\begin{aligned} CF(\Sigma', \beta', \alpha') &\rightarrow CF(\Sigma, \beta, \alpha), \\ \mathbf{y} &\mapsto (\mathbf{y}, x_0) \end{aligned}$$

is an isomorphism of chain complexes. Therefore, tensoring with x_0 gives an isomorphism

$$\Phi : SFH(-M', -\Gamma') \xrightarrow{\sim} SFH(-M, -\Gamma).$$

However, in order to show that the map is an identity morphism, we need to decompose the stabilization (i.e., attaching a handle to Σ' and adding α_0, β_0 to α', β') into a trivial stabilization and a sequence of handleslides. Let us consider the R_- case. (The R_+ case is left to the reader.) In this case, β_0 only intersects α_0 , but α_0 can intersect β'_i . If there are no other intersections, then we are done, since we have a trivial stabilization. Otherwise, consider the pushoff \bar{a}_0 of a_0 , obtained by isotoping the endpoints of a_0 along $\partial S'$, against the orientation of Γ' . If we stabilize $(S', R_+(\Gamma'), h')$ along \bar{a}_0 , then all the intersections between α_0 and β'_i will be eliminated, since the composition with the positive Dehn twist forces the arcs to go around the core of the attached 1-handle. In its place, if a'_{k+1} is the cocore of the 1-handle, then its image under the monodromy will intersect a_0 exactly once. Let us rename open books and assume $(S', R_+(\Gamma'), h')$ already has this property, namely we may assume that there is only one intersection between α_0 and $\cup_i \beta'_i$. The rest of the argument is identical to that of Lemma 5.6, and will be omitted.

6.1.3. Reduction to a sequence of trivial bypasses. Suppose now that (M', Γ') is a sutured submanifold of (M, Γ) , $M - M' = \partial M' \times [0, 1]$, $\partial M' = \partial M' \times \{0\}$, and the contact structure ξ on $\partial M' \times [0, 1]$ with convex boundary condition $\Gamma \cup \Gamma'$ is $[0, 1]$ -invariant. We now prove that there is an extension of $(S', R_+(\Gamma'), h')$ for (M', Γ') to $(S, R_+(\Gamma), h)$ for (M, Γ) , of the type constructed in Step 2 of Section 4, which can be decomposed into a sequence of trivial bypass attachments. The nature of this extension is such that it is obtained by adding “horizontal” Legendrian arcs of type $\delta \times \{t\} \subset \partial M' \times [0, 1]$ and “vertical” Legendrian arcs of type $\{p\} \times [0, t]$. We will see how the extension can be thought of as a sequence of trivial bypass additions.

Observe that, when we attach a neighborhood $N(d)$ of a trivial bypass arc d , then the result can be viewed more symmetrically as in Figure 9. (This we leave as an exercise for the reader.) This means that d can be viewed as the concatenation of three Legendrian arcs: two “vertical” arcs $\{p_1, p_2\} \times [0, t]$ and a “horizontal” arc $\delta \times \{t\}$, where δ connects p_1 and p_2 . In this subsection we make the assumption that all δ ’s, possibly with subscripts, do not intersect Γ' in the interior of δ , and all p_i ’s are in Γ' . Let c' be the component of $c - \Gamma'$ which is not part of ∂D_1 . Slide the endpoints of c' in the direction of Γ' if $c' \subset R_-(\Gamma')$ and in the direction of $-\Gamma'$ if $c' \subset R_+(\Gamma')$. We will call the resulting Legendrian arc \bar{a}_0 ; this notation agrees with the notation for the stabilizing

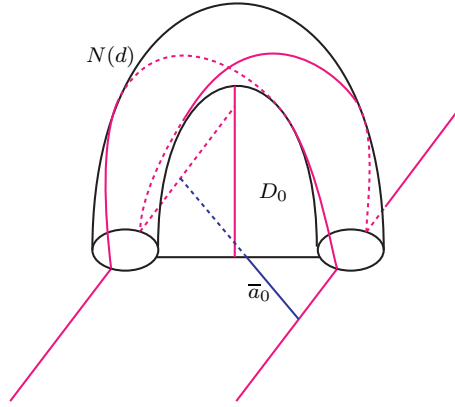


FIGURE 9. Attaching a trivial bypass arc d . We stabilize along \bar{a}_0 before attaching the bypass.

arc, used in Section 6.1.2. If we take a Legendrian-isotopic copy of \bar{a}_0 inside M' via an isotopy which fixes the endpoints, then we perform a stabilization as in Section 6.1.2 along the copy of \bar{a}_0 before attaching the bypass.

Now, if we have a Legendrian graph consisting of $\{p_1, p_2, p_3\} \times [0, t]$, together with $\delta_i \times \{t\}$, $i = 1, 2$, with $\partial\delta_i = \{p_i, p_{i+1}\}$, then attaching its standard Legendrian neighborhood is equivalent to attaching two bypass arcs as given in Figure 10; this is readily seen by sliding an endpoint of $\delta_2 \times \{t\}$ along the dividing set on the boundary of the union of M' and the neighborhood of the Legendrian arc $(\{p_1\} \times [0, t]) \cup (\delta_1 \times \{t\}) \cup (\{p_2\} \times [0, t])$.

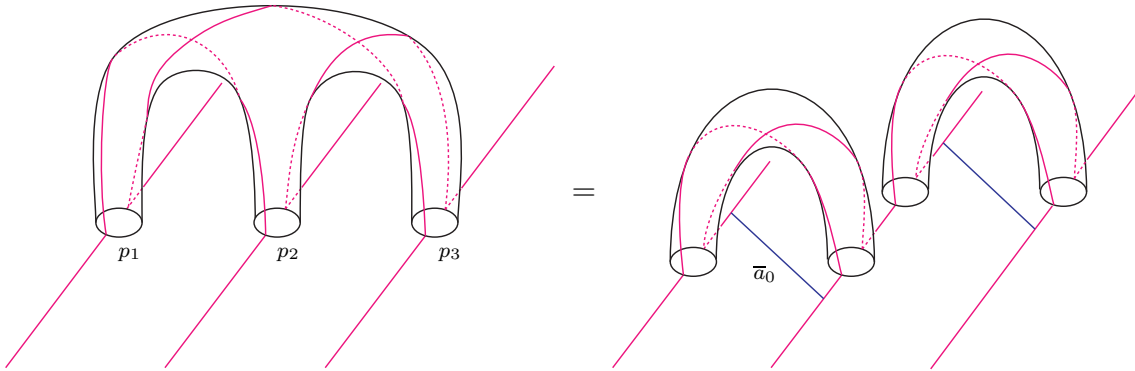


FIGURE 10. Sliding the bypass arc.

Finally, let γ_1 be a Legendrian arc given by the concatenation of $\{p_1, p_2\} \times [0, t]$ and $\delta_1 \times \{t\}$ with $\partial\delta_1 = \{p_1, p_2\}$, and we attach a Legendrian arc γ_2 consisting of $\{p_3\} \times [0, t]$ and $\delta_2 \times \{t\}$ with $\partial\delta_2 = \{p_3, q\}$, where q is an interior point of δ_1 . By sliding the endpoint of δ_2 , we see that attaching γ_1 and γ_2 is equivalent to attaching the two Legendrian arcs given in Figure 11. When attaching the first arc γ_1 , we first stabilize along \bar{a}_0 ; for the second arc γ_2 , the attachment of the first arc has the same effect as a stabilization.

Therefore, using the above trivial bypass arcs, we can construct a Legendrian graph L in $\partial M' \times [0, 1 - \varepsilon]$, which is the union of arcs of type $\{p\} \times [0, 1 - \varepsilon]$ and the 1-skeleton $L_{1-\varepsilon}$ of a cell

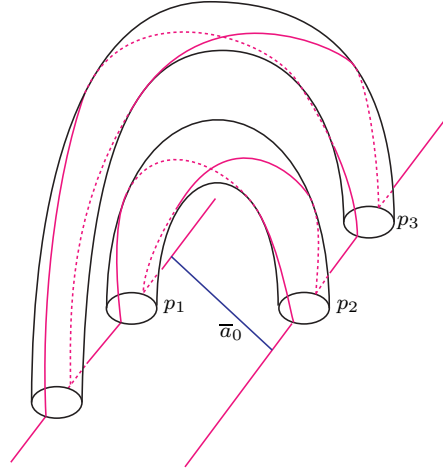


FIGURE 11.

decomposition of $\partial M' \times \{1 - \varepsilon\}$, each of whose cells have boundary with $tb = -1$. (Here $(p, 1 - \varepsilon)$ must lie in $L_{1-\varepsilon}$.) If we take the standard Legendrian neighborhood $N(L)$, then its complement $(\partial M' \times [0, 1 - \varepsilon]) - N(L)$ is also a standard neighborhood of a Legendrian graph K'' . The Legendrian neighborhood $N(L_{1-\varepsilon})$ can be enlarged via a contact isotopy so that $N(L_{1-\varepsilon})$ is $\partial M' \times [1 - 2\varepsilon, 1]$, with neighborhoods of Legendrian arcs of type $\{q\} \times [1 - 2\varepsilon, 1]$, $q \in \Gamma'$, removed. On the other hand, $N(\{p\} \times [0, 1 - \varepsilon])$ is viewed as a sufficiently thin/small neighborhood of the Legendrian arc $\{p\} \times [0, 1 - \varepsilon]$. The above description clearly indicates that the extension of K' to $K' \cup K''$ is an extension of the partial open book decomposition $(S', R_+(\Gamma'), h')$ to $(S, R_+(\Gamma), h)$ of the type described in Step 2 of Section 4. This completes the proof of Theorem 6.1.

7. A $(1 + 1)$ -DIMENSIONAL TQFT

In this section we describe a $(1 + 1)$ -dimensional TQFT, obtained by dimensional reduction. (Strictly speaking, the theory does not quite satisfy the TQFT axioms but has similar composition rules.)

7.1. Invariants of multicurves on surfaces. In this subsection we describe a TQFT-type invariant of a multicurve on a surface. Let Σ be a compact, oriented surface with nonempty boundary $\partial\Sigma$, and F be a finite set of points of $\partial\Sigma$, where the restriction of F to each component of $\partial\Sigma$ consists of an even number ≥ 2 of points. Part of the structure of a pair (Σ, F) is a labeling of each component of $\partial\Sigma - F$ by $+$ or $-$ so that crossing a point of F while moving along $\partial\Sigma$ reverses signs. Let $\#F = 2n$ be the cardinality of F . Also let K be a properly embedded, oriented 1-dimensional submanifold of Σ whose boundary is F and which divides Σ into R_+ and R_- in a manner compatible with the labeling of $\partial\Sigma - F$. As on $\partial\Sigma - F$, the sign changes every time K is crossed. Such a K will be called a *dividing set* for (Σ, F) .

We now list the properties satisfied by the TQFT.

TQFT Properties.

I. It assigns to each (Σ, F) a graded \mathbb{Z} -module $V(\Sigma, F)$. If Σ is connected, then

$$V(\Sigma, F) = \mathbb{Z}^2 \otimes \cdots \otimes \mathbb{Z}^2,$$

where the number of copies of \mathbb{Z}^2 is $r = n - \chi(\Sigma)$, and $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ is a graded \mathbb{Z} -module whose first summand has grading 1 and the second summand has grading -1 . We will refer to this grading as the *Spin^c-grading*. Moreover, if (Σ, F) is the disjoint union $(\Sigma_1, F_1) \sqcup (\Sigma_2, F_2)$, then

$$V(\Sigma_1 \sqcup \Sigma_2, F_1 \sqcup F_2) \simeq V(\Sigma_1, F_1) \otimes V(\Sigma_2, F_2).$$

II. To each K it assigns a subset $c(K) \subset V(\Sigma, F)$ of cardinality 1 or 2 of type $\{\pm x\}$, where $x \in V(\Sigma, F)$. If K has a homotopically trivial closed component, then $c(K) = \{0\}$.

III. Given (Σ, F) , let $\gamma, \gamma' \subset \partial\Sigma$ be mutually disjoint 1-dimensional submanifolds of $\partial\Sigma$, so that their endpoints do not lie in F . Suppose there is a diffeomorphism $\tau : \gamma \xrightarrow{\sim} \gamma'$ which sends $\gamma \cap F \xrightarrow{\sim} \gamma' \cap F$ and preserves the orientations on $\gamma \cap \partial\Sigma$ and $\gamma' \cap \partial\Sigma'$. If we glue (Σ, F) by identifying γ and γ' via τ , then the result will be denoted by (Σ', F') . Then there exists a map

$$\Phi_\tau : V(\Sigma, F) \rightarrow V(\Sigma', F'),$$

which satisfies

$$c(K) \mapsto c(\overline{K}),$$

where \overline{K} is obtained from K by gluing $K|_\gamma$ and $K|_{\gamma'}$. Here Φ_τ is well-defined up to an overall ± 1 multiplication.

See Figure 12 for an illustration of the gluing in Property III, when $(\Sigma, F) = (\Sigma'', F'') \sqcup (\Sigma''', F''')$, $K = K'' \sqcup K'''$, and γ, γ' are arcs. In this case, the gluing map is:

$$\Phi_\tau : V(\Sigma'', F'') \otimes V(\Sigma''', F''') \rightarrow V(\Sigma', F').$$

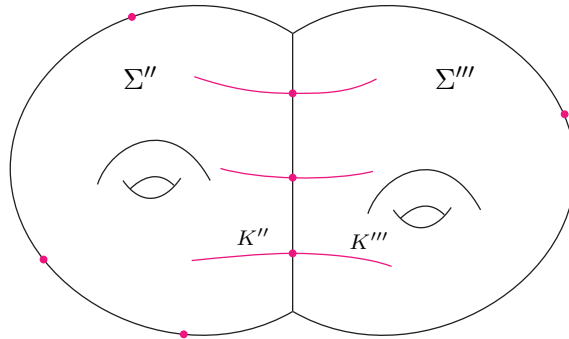


FIGURE 12. Gluing (Σ'', K'') and (Σ''', K''') . The red dots are F'' and F''' .

We will use the subscripts (i) to denote the Spin^c-grading: $V(\Sigma, F)_{(i)}$ is the graded piece with grading i and $\mathbb{Z}_{(i)}^n$ is the \mathbb{Z}^n -summand representing the i th graded piece.

Theorem 7.1. *There exists a nontrivial TQFT satisfying Properties I-III above.*

Proof. This TQFT arises by dimensional reduction of sutured Floer homology.

I. Given (Σ, F) , let $F_0 \subset \partial\Sigma$ be obtained from F by shifting slightly in the direction of $\partial\Sigma$. (We may think of points of F_0 as being situated halfway between successive points of F on $\partial\Sigma$.) We consider S^1 -invariant balanced sutured manifolds $(S^1 \times \Sigma, S^1 \times F_0)$, and let

$$V(\Sigma, F) = SFH(-(S^1 \times \Sigma), -(S^1 \times F_0)).$$

The reason for using F_0 instead of F in the definition is explained below in II when the role of contact structures is explained. The Spin^c -grading for $V(\Sigma, F)$ corresponds to the different relative Spin^c -structures on $(S^1 \times \Sigma, S^1 \times F_0)$.

The next lemma determines $V(\Sigma, F)$, up to isomorphism.

Lemma 7.2. *If Σ is connected, then*

$$SFH(-(S^1 \times \Sigma), -(S^1 \times F_0)) \simeq (\mathbb{Z}_{(-1)} \oplus \mathbb{Z}_{(1)})^{\otimes r},$$

where $r = n - \chi(\Sigma)$.

Proof. This follows from Juhász' tensor product formula [Ju2, Proposition 8.10] for splitting sutured manifolds along product annuli, together with the observation that when $n = 2$ and $\Sigma = D^2$, we have $SFH(-(S^1 \times D^2), -(S^1 \times F_0)) \simeq \mathbb{Z}_{(-1)} \oplus \mathbb{Z}_{(1)}$, split according to the relative Spin^c -structure. (See [HKM3, Section 5, Example 2].) \square

Finally, the property

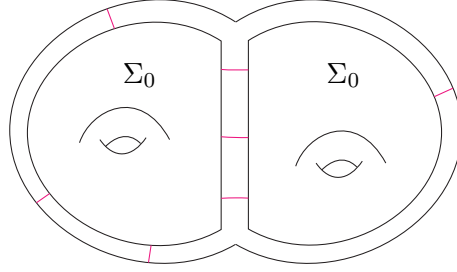
$$V(\Sigma_1 \sqcup \Sigma_2, F_1 \sqcup F_2) \simeq V(\Sigma_1, F_1) \otimes V(\Sigma_2, F_2)$$

is immediate from the definition of the sutured Floer homology groups.

II. Next, there is a 1 – 1 correspondence between dividing sets K of (Σ, F) without homotopically trivial closed curves and tight contact structures with boundary condition $(S^1 \times \Sigma, S^1 \times F_0)$, up to isotopy rel boundary. For the correspondence to hold we require that $\partial\Sigma \neq \emptyset$. The map from dividing sets to contact structures is easy: simply consider the S^1 -invariant contact structure ξ_K on $S^1 \times \Sigma$ so that the dividing set on each $\{pt\} \times \Sigma$ is $\{pt\} \times K$. It was shown in [Gi3, H2] that the map, when restricted to the subset of dividing sets K without homotopically trivial curves, gives a bijection with the set of isotopy classes of tight contact structures on $(S^1 \times \Sigma, S^1 \times F_0)$. Now, to each K we assign $EH(\xi_K) \subset SFH(-(S^1 \times \Sigma), -(S^1 \times F_0))$. If K has a homotopically trivial curve, then ξ_K is overtwisted, and $EH(\xi_K) = \{0\}$.

Finally we explain why we use F_0 instead of F in $(S^1 \times \Sigma, S^1 \times F_0)$. The dividing set $S^1 \times F_0$ of $\partial(S^1 \times \Sigma)$ does not intersect the dividing set of $\{pt\} \times \Sigma$, since the two surfaces are transverse. This means that F_0 must lie between the endpoints F of K .

III. This is a corollary of Theorem 1.1: In order to apply Theorem 1.1, slightly shrink Σ to Σ_0 inside the glued-up surface Σ' . See Figure 13. If we write $\Sigma - \Sigma_0 = \partial\Sigma \times [0, 1]$ with $\partial\Sigma \times \{0\} = \partial\Sigma_0$ and $\partial\Sigma \times \{1\} = \partial\Sigma$, then the dividing set K_0 on $\Sigma' - \Sigma_0$ is obtained from $F \times [0, 1]$ by identifying $F|_\gamma \times \{1\}$ with $F|_{\gamma'} \times \{1\}$ via ϕ . Let ξ_{K_0} be the S^1 -invariant contact structure on $S^1 \times (\Sigma' - \Sigma_0)$ corresponding to the dividing set K_0 . The contact structure ξ_{K_0} induces the map $\Phi_{\xi_{K_0}} = \Phi_\tau$ from $V(\Sigma, F)$ to $V(\Sigma', F')$. This completes the proof of Theorem 7.1. \square

FIGURE 13. The dividing set K_0 on $\Sigma' - \Sigma_0$ is given in red.

Remark 7.3. *There is another grading for $V(\Sigma, F)$, a relative grading called the Maslov grading, which is largely invisible for the time being since all the generators have the same Maslov grading.*

7.2. Analysis of $\Sigma = D^2$. Suppose $\Sigma = D^2$ and F consists of $2n$ points on ∂D^2 . In this case, the set of dividing sets K without closed components corresponds to the set of crossingless matchings of F . A *crossingless matching* of F is a collection of n properly embedded arcs in D^2 with endpoints on F so that each endpoint is used once and no two arcs intersect in D^2 . The orientation condition is trivially satisfied for a crossingless matching. If K has a closed component, then the component must be homotopically trivial. Thus the corresponding contact structure ξ_K is overtwisted, and $c(K) = \{0\}$.

n=1. When $n = 1$, $V(\Sigma, F) = \mathbb{Z}_{(0)}$, which is generated by the unique K which connects the two points. (By this we mean \mathbb{Z} is generated by either element of $c(K)$.)

n=2. When $n = 2$, $V(\Sigma, F) = \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(-1)}$. We claim that $V(\Sigma, F)$ is generated by $c(K_+)$ and $c(K_-)$, given as in Figure 14. Here K_+ and K_- are the two dividing sets, both ∂ -parallel. The

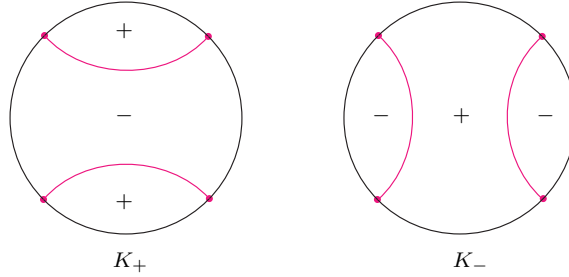
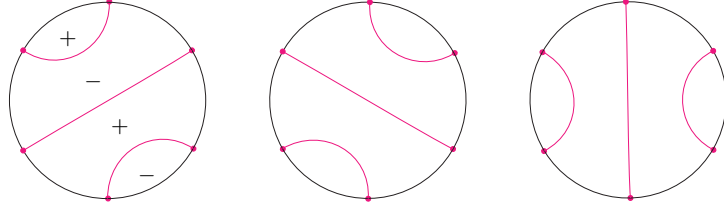


FIGURE 14.

grading for $c(K)$ can be calculated by taking $\chi(R_+) - \chi(R_-)$, where R_+ (resp. R_-) is the positive (resp. negative) region of $\Sigma - K$. Hence the degrees are 1 and -1 for K_+ and K_- , respectively. As calculated in [HKM3, Section 5, Example 3], there is a Heegaard diagram for which the EH class for ξ_{K_+} is the unique tuple representing its Spin^c -structure (and similarly for K_-). Hence $c(K_+)$ generates the first summand and $c(K_-)$ generates the second summand, with respect to any coefficient system.

n=3. When $n = 3$, $V(\Sigma, F)$ decomposes into $\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(0)}^2 \oplus \mathbb{Z}_{(-2)}$. The first and last summands are generated by $c(K)$ for ∂ -parallel K . The middle $\mathbb{Z}_{(0)}^2$ must support three configurations K_1, K_2, K_3 . See Figure 15.

FIGURE 15. K_1, K_2, K_3 , from left to right.

We have the following:

Lemma 7.4. *The sets $c(K_1), c(K_2), c(K_3)$ are nonzero and distinct. Moreover, their elements are primitive.*

Over $\mathbb{Z}/2\mathbb{Z}$, the lemma implies that $c(K_1) + c(K_2) = c(K_3)$, i.e., $c(K_3)$ is a superposition of the other two states $c(K_1)$ and $c(K_2)$.

Proof. Consider an arc $\gamma \subset \partial\Sigma$ with $\#(F \cap \gamma) = 2$. Take a disk $\Sigma' = D^2$ with $\#F' = 2$, and pick an arc $\gamma' \subset \partial\Sigma'$ with $\#(F' \cap \gamma') = 2$. Then attach Σ' onto Σ so that γ and γ' are identified and $F'' = F \cup F' - \gamma$ satisfies $\#F'' = 4$. Observe that the \mathbb{Z} -module $V(\Sigma', F') \simeq \mathbb{Z}$ is generated by a unique element K' , which is a ∂ -parallel arc. Label the points of F in clockwise order from 1 to 6, so that 1 is 2pm, 2 is 4pm, etc., and let $\Phi_j, j = 1, 2, 3$, be the gluing map

$$V(\Sigma, F) \rightarrow V(\Sigma \cup \Sigma', F''),$$

obtained by attaching the ∂ -parallel arc K' from j to $j + 1$. It sends $c(K_i) \mapsto c(K_i \cup K')$. See Figure 16. Restricted to $\mathbb{Z}_{(0)}^2$, the image of Φ_j is one of the two summands $\mathbb{Z}_{(1)}$ or $\mathbb{Z}_{(-1)}$. Hence we view Φ_j as a map $\mathbb{Z}_{(0)}^2 \rightarrow \mathbb{Z}_{(\pm 1)}$.

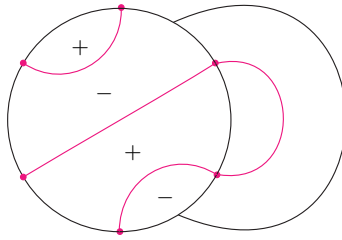


FIGURE 16. The diagram represents $\Phi_1(c(K_1))$. K_1 is the crossingless matching inside the circle, and the ∂ -parallel arc, representing a generator of $V(\Sigma', F') = \mathbb{Z}$ with $\#F' = 2$, is attached from 1 to 2.

Suppose $K_i \cup K'$ does not have a closed component; there is always some Φ_j for which this is true. Then we have reduced to the case $n = 2$, where we already know that each representative of $c(K_i \cup K')$ is nonzero and primitive. Since $\Phi_j : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ maps $c(K_i) \mapsto c(K_i \cup K')$ and the latter is primitive, it follows that $c(K_i)$ must also be primitive.

Next, $c(K_i \cup K') = EH(\xi_{K_i \cup K'}) = \{0\}$ if $K_i \cup K'$ has a closed (and necessarily homotopically trivial) component. Hence, by attaching Σ' at the appropriate locations (i.e., checking which Φ_1, Φ_2 or Φ_3 annihilates $c(K_i)$), we can determine the locations of all the ∂ -parallel (or outermost)

arcs of $K_i \subset \Sigma$. Since the location of the ∂ -parallel arcs determines K_i , it follows that the $c(K_i)$ must be distinct. \square

By inductively applying the above procedure, we obtain the following:

Proposition 7.5. *All crossingless matchings K of $(\Sigma = D^2, F)$ with $\#F = 2n$ are distinguished by $c(K) \subset V(\Sigma, F)$ and are primitive. Equivalently, all the tight contact structures on $S^1 \times D^2$, $\#F = 2n$, are distinguished by their contact invariant in $SFH(-(S^1 \times D^2), -(S^1 \times F_0))$.*

The proof is left to the reader. Lemma 7.4 and its generalization Proposition 7.5 are rather surprising, since the dimension of $V(D^2, F)$ with $\#F = 2n$ is 2^{n-1} , whereas the number of crossingless matchings on (D^2, F) is the *Catalan number* $C_n = \frac{1}{n+1} \binom{2n}{n}$, which is greater than or equal to 2^{n-1} , and grows roughly twice as fast as a function of n . This means that all the $c(K)$'s are “tightly packed” inside $V(D^2, F)$, especially when the coefficient ring is $\mathbb{Z}/2\mathbb{Z}$.

Also recall that the dimensions of our $V(D^2, F)$ with $\#F = 2n$ are the same as that of $(1+1)$ -dimensional, level $k = 2$, $sl(2, \mathbb{C})$ TQFT. It would be interesting to compare the two TQFT's.

7.3. The ± 1 ambiguity over \mathbb{Z} -coefficients. In this subsection we prove Theorem 7.6 and deduce from it that the ± 1 ambiguity of the contact invariant $EH(M, \Gamma, \xi)$ in $SFH(-M, -\Gamma)$ over \mathbb{Z} cannot be removed and that the gluing map Φ_ξ of Theorem 1.1 is well-defined *only up to* an overall \pm sign over \mathbb{Z} . This proves Theorem 1.4, stated in the Introduction.

Theorem 7.6. *There is no single-valued lift of $c(K) \subset V(\Sigma, F)$ for all K, Σ, F , with \mathbb{Z} -coefficients.*

Proof. Assume the invariants of dividing curves are single-valued. Consider the example $\Sigma = D^2$ and $\#F = 6$. Recall K_1, K_2, K_3 from Figure 15. By Lemma 7.4, each element of $c(K_i)$, $i = 1, 2, 3$, is primitive in $\mathbb{Z}_{(0)}^2$. We also use the same maps $\Phi_j : \mathbb{Z}_{(0)}^2 \rightarrow \mathbb{Z}^2$, $j = 1, 2, 3$.

We compute the following:

$$\begin{aligned} \Phi_1 : c(K_1) &\mapsto c(K_+), \quad c(K_2) \mapsto c(K_+), \quad c(K_3) \mapsto 0, \\ \Phi_2 : c(K_1) &\mapsto 0, \quad c(K_2) \mapsto c(K_-), \quad c(K_3) \mapsto c(K_-), \\ \Phi_3 : c(K_1) &\mapsto c(K_+), \quad c(K_2) \mapsto 0, \quad c(K_3) \mapsto c(K_+). \end{aligned}$$

Here $c(K_+)$ and $c(K_-)$ are generators of $V(D^2, F')$ with $\#F' = 4$. Since the image of each $\Phi_j(\mathbb{Z}^2)$ is \mathbb{Z} , generated by either $c(K_+)$ or $c(K_-)$, we view Φ_j as a map $\mathbb{Z}^2 \rightarrow \mathbb{Z}$.

Let us analyze Φ_1 in more detail. Write $c(K_1)$ as $(1, 0) \in \mathbb{Z}^2$, since it is primitive. Then $\Phi_1 : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ maps $(1, 0) \mapsto 1$. We can then decompose \mathbb{Z}^2 into $\mathbb{Z} \oplus \mathbb{Z}$ so that $(0, 1)$ generates $\ker \Phi_1$, possibly after an appropriate isomorphism of \mathbb{Z}^2 . Without loss of generality, $c(K_3) = (0, 1)$. Since $\Phi_1 : c(K_2) \mapsto 1$, it follows that $c(K_2) = (1, a)$, $a \in \mathbb{Z}$.

Next consider Φ_2 . Since $(1, 0) \mapsto 0$, $(0, 1) \mapsto 1$, and $(1, a) \mapsto 1$, it follows that $a = 1$.

Finally, Φ_3 maps $(1, 0) \mapsto 1$, $(0, 1) \mapsto 1$, and should map $(1, 1) \mapsto 2$, but instead sends it to 0, a contradiction. \square

Theorem 7.7. *± 1 monodromy exists in sutured Floer homology. That is, there is a sequence of stabilization, destabilization, handleslide, and isotopy maps which begins and ends at the same configuration, so that their composition is $-id$.*

Proof. When working over \mathbb{Z} , $EH(M, \Gamma, \xi)$ and Φ_ξ are defined up to a factor of ± 1 . The only reason for the introduction of this factor was the *possibility* of the existence of ± 1 monodromy. Since single-valued lifts do not always exist by Theorem 7.6, it follows that ± 1 monodromy must exist. \square

This proof is unsatisfying in the sense that it does not explain the root cause of the existence of monodromy, nor does it give a specific sequence of maps which exhibits nontrivial monodromy.

Question 7.8. *Is there monodromy in Heegaard Floer homology, i.e., when the 3-manifold is closed? In particular, is there an explicit sequence of stabilization, destabilization, handleslide, and isotopy maps which begins and ends at the same configuration, so that their composition is $-id$ for $\widehat{HF}(S^1 \times S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$?*

7.4. A useful gluing isomorphism. In this subsection we give a useful gluing map and explore some consequences.

Let γ be a properly embedded arc on Σ which is transverse to K and intersects K exactly once. Suppose we cut (Σ, F) and K along γ to obtain (Σ', F') and K' . This is the reverse procedure of gluing (Σ', F') and K' along disjoint subarcs $\gamma', \gamma'' \subset \partial\Sigma'$, where each arc intersects F' exactly once. We then have:

Lemma 7.9. *The gluing map $\Phi : V(\Sigma', F') \rightarrow V(\Sigma, F)$ is an isomorphism.*

If γ decomposes Σ into two components (Σ'', F'') and (Σ''', F''') , then the gluing map is:

$$\Phi : V(\Sigma'', F'') \otimes V(\Sigma''', F''') \xrightarrow{\sim} V(\Sigma, F).$$

Proof. We interpret the gluing map Φ as a gluing map $\Phi_0 : V(\Sigma', F') \rightarrow V(\Sigma, F)$, where the gluing occurs along a ∂ -parallel convex annulus A as given in Figure 17.

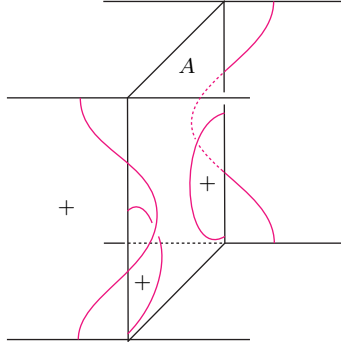


FIGURE 17. The top and bottom of the annulus are identified.

First we prove that $\Phi = \Phi_0$. Let $(M, \Gamma) = (S^1 \times \Sigma, S^1 \times F_0)$, where F_0 is the pushoff of F in the direction of $\partial\Sigma$. Also let (M', Γ') be the sutured manifold obtained from $(S^1 \times \Sigma', S^1 \times F'_0)$ by slightly retracting Σ' to Σ'_0 ; here F'_0 is the pushoff of F' . Let ξ_0 be the contact structure on $M - \text{int}(M')$, given as the union of the invariant contact structures on a neighborhood of $\partial M'$ with dividing set $S^1 \times F'_0$ and on a neighborhood of A with ∂ -parallel dividing set. Since the dividing set on $\partial(M - \text{int}(M'))$ is of the type $S^1 \times \{\text{finite set}\}$, ξ_0 is an S^1 -invariant contact structure by [Gi3, H2], and is encoded by the “minimal” dividing set K_0 on $\Sigma - \Sigma'_0$.

We now briefly sketch why $\xi_{K'} \cup \xi_0$ is isotopic to ξ_K . Let γ_1, γ_2 be the components of Γ which intersect ∂A and $\delta_1, \dots, \delta_m$ be the components of Γ which do not intersect ∂A . For each δ_i , there is a parallel copy δ'_i on $\partial M'$. Moreover, there is a Legendrian arc from δ_i to δ'_i which has zero twisting number with respect to a surface parallel to $\Sigma - \Sigma'_0$. For each γ_i , there are two components γ'_i and γ''_i on $\partial M'$ which share a parallel arc with γ_i . Hence there are Legendrian arcs from γ_i to γ'_i and from γ_i to γ''_i which have zero twisting number as well. The above Legendrian arcs constrain K_0 so that $K' \cup K_0$ is isotopic to K . This proves $\Phi = \Phi_0$.

We next prove that Φ_0 is an isomorphism. According to Juhász [Ju2], gluing along a product annulus gives an isomorphism of sutured Floer homology groups. Although our situation is slightly different, the result is the same. By [HKM3, Theorem 6.2], $V(\Sigma', F')$ is a direct summand of $V(\Sigma, F)$ since the dividing set on A is ∂ -parallel. Now, according to Proposition 6.4, Φ_0 is indeed the direct summand map of [HKM3, Theorem 6.2]. More precisely, Φ_0 induces an isomorphism onto the Spin^c -direct summand corresponding to the ∂ -parallel dividing set with relative half-Euler class $\chi(R_+) - \chi(R_-) = 2 - 0 = 2$. To see that $V(\Sigma', F') \simeq V(\Sigma, F)$ under the map Φ_0 , we use a rank argument. Both $V(\Sigma', F')$ and $V(\Sigma, F)$ are isomorphic to \mathbb{Z}^r , where $r = \frac{1}{2}(\#F) - \chi(\Sigma)$. Since $V(\Sigma', F')$ is a direct summand of $V(\Sigma, F)$ and they are both free with the same rank, it follows that $V(\Sigma', F') \simeq V(\Sigma, F)$. \square

As an application of Lemma 7.9, we give a sufficient condition for a dividing set K for (Σ, F) to have $c(K)$ which is nonzero and primitive in $V(\Sigma, F)$, when \mathbb{Z} -coefficients are used. A connected component of $\Sigma - K$ which is not connected to $\partial\Sigma$ is called an *isolated region of K in Σ* . We say that K is *isolating* if there is an isolated region of K in Σ , and *nonisolating* if there is no isolated region. For example, if K has a homotopically trivial closed curve, then it is isolating.

We then have the following:

Proposition 7.10. *With \mathbb{Z} -coefficients, the dividing set K has nonzero and primitive $c(K)$ if K is nonisolating.*

Proof. Suppose Σ is connected. (If Σ is not, we consider each component of Σ separately.) If $(\Sigma, F) = (D^2, F)$, then we are done by Proposition 7.5. Therefore, suppose $\Sigma \neq D^2$. In view of Lemma 7.9, it suffices to find a properly embedded arc $\gamma \subset \Sigma$ which intersects K exactly once, so that cutting along it increases the Euler characteristic of Σ by one. Let Σ_0 be a connected component of $\Sigma - K$ which has Euler characteristic $\neq 1$. Since K is nonisolating, Σ_0 must nontrivially intersect $\partial\Sigma$. It is then easy to find a properly embedded arc $\gamma_0 \subset \Sigma$ which lies in Σ_0 , and which is not ∂ -parallel in Σ_0 . We can isotop the endpoints of γ_0 along $\partial\Sigma$ so the resulting γ intersects K exactly once. \square

We also have the following corollary of Lemma 7.9:

Proposition 7.11. *With \mathbb{Z} -coefficients, $V(\Sigma, F)$ is generated $c(K)$, where K ranges over all dividing sets for which $\partial K = F$.*

Proof. The assertion is clearly true when $\Sigma = D^2$ and $\#F = 2$ or 4 . Now, any (Σ, F) can be split along an arc γ so that the resulting (Σ', F') satisfies $\chi(\Sigma') = \chi(\Sigma) + 1$ and $\#F' = \#F + 2$, and so that $V(\Sigma', F') \simeq V(\Sigma, F)$. Once we reach $\Sigma' = D^2$, a good choice of splitting will decrease $\#F'$ of each component, until each component is (D^2, F) with $\#F = 2$ or 4 . The proposition follows by gluing. \square

7.5. Analysis when Σ is an annulus. Suppose Σ is an annulus. We consider the situation where F consists of two points on each boundary component. The calculations will be done in \mathbb{Z} -coefficients, but calculations in a twisted coefficient system will certainly yield more information. See for example [GH].

By Juhász' formula, $V(\Sigma, F) = \mathbb{Z}^2 \otimes \mathbb{Z}^2 = \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(0)}^2 \oplus \mathbb{Z}_{(-2)}$. One can easily see that $\mathbb{Z}_{(2)}$ is generated by a ∂ -parallel K_+ with two positive ∂ -parallel arcs, and $\mathbb{Z}_{(-2)}$ is generated by a ∂ -parallel K_- with two negative ∂ -parallel arcs.

It remains to analyze $\mathbb{Z}_{(0)}^2$. The nonisolating dividing sets K with nontrivial $c(K) \subset \mathbb{Z}_{(0)}^2$ are the following: K'_0 and K'_1 , which have two ∂ -parallel arcs of opposite sign and one closed curve, L_0 consisting of two parallel arcs from one boundary component to the other, as well as L_j , obtained from L_0 by performing j positive Dehn twists about the core curve of the annulus. See Figure 18. The other possible dividing sets K , besides those with homotopically trivial components, have at least two parallel closed curves. The corresponding contact structure will necessarily have at least 2π -torsion. It was proved in [GHV] that any contact structure with 2π -torsion has vanishing contact invariant over \mathbb{Z} .

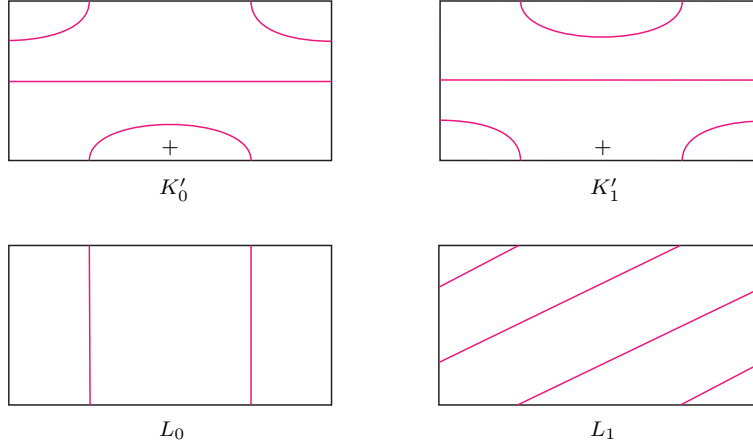


FIGURE 18. The sides of each annulus are identified.

First consider the map $\Phi : \mathbb{Z}_{(0)}^2 \rightarrow \mathbb{Z}_{(2)}$, obtained by attaching an annulus with configuration K_+ from below. Since a homotopically trivial curve is created, we have $\Phi(c(K'_0)) = \{0\}$. Also, $\Phi(c(K'_1)) = \{0\}$, since the resulting dividing set will have two parallel closed curves. On the other hand, the $c(L_i)$ all map to the generator $c(K_+)$ of $\mathbb{Z}_{(2)}$. Hence the map Φ is surjective, and must have $\ker \Phi \simeq \mathbb{Z}$. Next, since K'_0 and K'_1 are nonisolating, $c(K'_0)$ and $c(K'_1)$ must be primitive; this implies that $c(K'_0) = c(K'_1)$ and generate $\ker \Phi$. Now, one can make a coordinate change if necessary so that $c(L_0) = \{\pm(1, 0)\}$ and $c(K'_i) = \{\pm(0, 1)\}$.

Next we compute $c(L_1)$. For this, we use Lemma 7.4 and the following fact which follows from the proof of Theorem 7.6: for $\Sigma = D^2$ and $n = 3$, a representative $\bar{c}(K_3)$ of $c(K_3)$ is a superposition of representatives $\bar{c}(K_1)$ and $\bar{c}(K_2)$ of $c(K_1)$ and $c(K_2)$ with ± 1 coefficients. Observe that K_1 is obtained from K_3 by applying a bypass attachment from the front, and K_2 is obtained from K_3 by a bypass attachment to the back. It is easy to see from the Φ in the previous paragraph that $c(L_1) = \{\pm(1, n)\}$ for some integer n . Given the configuration K'_0 , take a bypass arc of attachment δ with endpoints on the two ∂ -parallel arcs and one other intersection point with K'_0 , namely

along the closed component. Take a small disk D^2 about δ . Consider the gluing map

$$\Psi : V(D^2, K'_0|_{\partial D^2}) \otimes V(\Sigma - D^2, K'_0|_{\partial \Sigma - D^2} \cup F) \rightarrow V(\Sigma, F).$$

By tensoring the $\bar{c}(K_i)$ with $\bar{c}(K'_0|_{\Sigma - D^2})$, the equation $\bar{c}(K_3) = \pm \bar{c}(K_1) \pm \bar{c}(K_2)$ becomes

$$\bar{c}(K'_0) = \pm \bar{c}(L_0) \pm \bar{c}(L_1).$$

This means $(0, 1) = \pm(1, 0) \pm (1, n)$. The only possible solutions are $(0, 1) = (1, 0) - (1, -1)$ or $(0, 1) = -(1, 0) + (1, 1)$. (The two possibilities are equivalent after a basis change.) Hence $c(L_1) = \{\pm(1, 1)\}$, for example.

In general, we conjecture that $c(L_n) = \{\pm(1, n)\}$. A proof of this conjecture requires a more careful sign analysis than we are willing to do for the moment.

7.6. Determination of nonzero elements $c(K)$ in $V(\Sigma, F)$. In this section we prove Theorem 1.5, i.e., we determine exactly which elements K have nonzero invariants $c(K)$ in $V(\Sigma, F)$ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients.

Proposition 7.12. *If K is isolating, then $c(K) = 0$ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients.*

Proof. Suppose first that there is an isolated region Σ_0 which is an annulus. In that case, take an arc of attachment δ of a bypass which intersects the two boundary components of Σ_0 , and some other component of K , in that order. By the TQFT property applied to a small neighborhood D of δ and $\Sigma - D$, we see that if K' (resp. K'') is obtained from K by applying a bypass from the front (resp. bypass to the back), then $c(K) = c(K') + c(K'')$, since the corresponding fact is true on D . One easily sees that K' and K'' are isotopic, and is K with $\partial\Sigma_0$ removed. With $\mathbb{Z}/2\mathbb{Z}$ -coefficients, then, $c(K) = 2c(K') = 0$.

Next suppose that Σ_0 has more than one boundary component, and is outermost among all isolated regions, in the sense that one boundary component γ of Σ_0 is adjacent to a component Σ_1 whose boundary intersects $\partial\Sigma$. Also suppose that Σ_0 is not an annulus. Take an arc of attachment δ which begins on γ , intersects γ after traveling inside Σ_0 , and ends on an arc component of K on $\partial\Sigma_1$. Choose δ so that $\Sigma_0 - \delta$ has two components, one which is an annulus and the other which has Euler characteristic $> \chi(\Sigma_0)$. Then apply the bypass attachments from the front and to the back to obtain K' , K'' as in the previous paragraph. Now, $c(K) = c(K') + c(K'')$, and one of $c(K')$ or $c(K'')$ is zero, since it possesses an annular isolated region. This reduces the number of components of $\partial\Sigma_0$.

Finally suppose that $\partial\Sigma_0$ is connected. If Σ_0 bounds a surface of genus $g > 1$, then the above procedure can split $c(K) = c(K') + c(K'')$, where both $c(K')$ and $c(K'')$ have isolated regions with connected $\partial\Sigma_0$ and strictly smaller genus. Hence suppose that Σ_0 bounds a once-punctured torus. Also, by cutting along arcs as in Proposition 7.10, we may assume that Σ itself is a once-punctured torus with one ∂ -parallel arc and one closed curve parallel to the boundary. Choose δ as given in Figure 19, namely, δ begins on the ∂ -parallel arc and intersects $\partial\Sigma_0$ twice, and restricts to a nontrivial arc on Σ_0 . The resulting K' and K'' are the center and right-hand diagrams. Now cut along the properly embedded, non-boundary-parallel arc τ which intersects each of K' and K'' exactly once. Applying Lemma 7.9, we see that $c(K') = c(K'')$ if and only if the cut-open dividing curves K'_0 and K''_0 have equal invariants in the cut-open surface. Finally, observe that, on the cut-open surface (an annulus), $c(K'_0) = c(K''_0)$ since they correspond to K_0 and K_1 , discussed in Subsection 7.5. \square

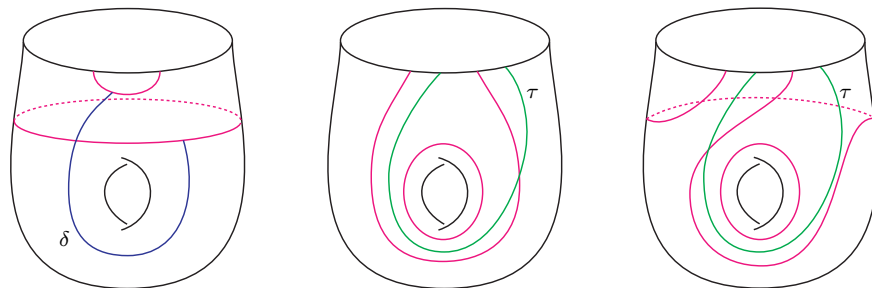


FIGURE 19.

Propositions 7.12 and 7.10, together give Theorem 1.5.

In the case of \mathbb{Z} -coefficients we expect the following to hold:

Conjecture 7.13. *Over \mathbb{Z} -coefficients, the following are equivalent:*

- (1) $c(K) \neq 0$;
- (2) $c(K)$ is primitive;
- (3) K is nonisolating.

The difficulty comes from not being able to determine whether $c(K)$ is divisible by 2 with \mathbb{Z} -coefficients, which in turn stems from our ± 1 difficulty in Subsection 7.3. When twisted coefficients are used, the result is quite different, and will yield substantially more information [GH].

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